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On the dynamics of generic non-Abelian free actions

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Abstract. We investigate some global generic properties of the dynamics associated to non-Abelian free actions in certain special cases. The main properties considered in this paper are related to the existence of dense orbits, to ergodicity and to topological rigidity. We first deal with them in the case of conservative homeomorphisms of a manifold and C^1-diffeomorphisms of a surface. Groups of analytic diffeomorphisms of a manifold which, in addition, contain a Morse-Smale element and possess a generating set close to the identity are considered as well. From our discussion we also derive the existence of a rigidity phenomenon for groups of skew-products which is opposed to the phenomenon present in Furstenberg's celebrated example of a minimal diffeomorphism that is not ergodic (cf. [Ma]).

Keywords: free groups, dense orbits, vector fields.

Mathematical subject classification: 37A15, 37B05, 37C20.

1 Introduction

The dynamics generated by one "generic" diffeomorphisms on a compact manifold M is an intensively studied subject. This study is basically divided into the case of general C^r-diffeomorphisms ($r = 0, 1, \dots, \infty, \omega$) and the case of diffeomorphisms preserving a fixed volume on M (some works are also devoted to symplectic diffeomorphisms). Whereas this is undoubtedly an important topic, it cannot reflect the typical behavior of more general dynamics as already pointed out by Gromov [Gr1] and explained below.

A dynamical system should be understood as a *group action* i.e. it is obtained through a faithful representation of an abstract group G into $\mathrm{Diff}^r(M)$. Therefore the traditional case of the dynamics generated by one diffeomorphisms corresponds to a $\mathbb{Z}$-action. In this broader sense, it soon becomes clear that there

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are many more dynamical systems (i.e. group actions) than those associated to $\mathbb{Z}$-actions which suggests that the typical features of $\mathbb{Z}$-actions may be different from the features of group actions "chosen at random". The present work is a first attempt of making sense and investigating some of the generic properties of a "dynamics" in this general sense.

In this paper we focus attention on non-Abelian free group actions which we believe are likely to lead to a fair idea of typical dynamics in our sense. In fact, there are at least three reasons to consider these actions. First, as observed in Section 5, a collection of diffeomorphisms "randomly chosen" generates a non-Abelian free group. Secondly, considering the space of abstract groups, one can wonder which are the "generic properties" of a group, so as to try to find good representatives for this space. This question was first considered in [Gr2] (see also [Ch]) and it turned out that, in a reasonable sense, a typical group is word-hyperbolic (but it is not a finite extension of $\mathbb{Z}$ which also confirms that $\mathbb{Z}$ is too particular as a group). It is then natural to consider a non-Abelian free group as a good representative of the space of groups which can act on a manifold since these hyperbolic groups always contain non-Abelian free groups on (say) two generators (conversely a free group on two or more generators is clearly word-hyperbolic, cf. [Gr2]). At last, dealing with free groups, we do not become involved with "functional identities" coming from relations in the group which would quickly put the problem out of reach.

In view of the preceding, we shall be concerned with non-Abelian free actions. Among the several dynamical features which could be analysed, we concentrate our efforts on 3 aspects of global nature: density of orbits, ergodicity and structural stability. As it will be clear after Section 2, the investigation of these questions depend on the nature of the generators (i.e. they depend on specific open sets of $\mathrm{Diff}^r(M)$ to which the generators are supposed to belong).

This work consists of 2 parts which are rather different in their settings. However these two settings are both natural, at least for the beginning of such discussion. The first part deals with volume-preserving diffeomorphisms and requires low-regularity for the diffeomorphisms involved. The second part requires real analycity of these diffeomorphisms, is not volume-preserving and the generators of the group in question are supposed to be "close to the identity". For these analytic actions, we shall establish a phenomenon of rigidity which may be thought of as being "opposite" to the structural stability. Actually this phenomenon seems to appear with significative frequency for actions of free groups while it does not take place for $\mathbb{Z}$-actions. Curiously enough key arguments of these parts have some similarity between them; in fact they are both based on controlling the dynamics on a small region by comparing it with a vector field. We then try to globalize this "local" dynamics as far as it is possible. At the end of this article we also raise a few questions related to our main results.

2 Statement of results

Let us begin by describing an example of a free action which will imply that most assumptions in our statements cannot be dropped. Let Γ be the free group generated by two Mœbius transformations $\mathcal{M}_1$, $\mathcal{M}_2$ where the projective action of $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) on the Riemann sphere $\mathbb{C}P(1) \simeq S^2$ is given by a "south pole - north pole" (resp. "west pole - east pole") diffeomorphism. When $\mathcal{M}_1$, $\mathcal{M}_2$ are chosen so as to define a Schottky group, the action of Γ on S^2 ($\simeq \mathbb{C}P(1)$) possesses an invariant Cantor set (which coincides with the Limit set of Γ). Furthermore, this action is structurally stable: if f_1 (resp. f_2) is a C^1-diffeomorphism of S^2 which is C^1-close to the diffeomorphism induced by $\mathcal{M}_1$ (resp. $\mathcal{M}_2$), then there is a homeomorphism $\mathsf{h} : S^2 \to S^2$ such that $f_1 \circ \mathsf{h} = \mathsf{h} \circ f_1$ and $f_2 \circ \mathsf{h} = \mathsf{h} \circ f_2$. In other words, the group generated by f_1, f_2 is topologically conjugate to Γ.

In our example, the diffeomorphisms induced by $\mathcal{M}_1$, $\mathcal{M}_2$ are Morse-Smale diffeomorphisms whose individual dynamics is very simple: apart from the repelling fixed point, all points converge to the attracting fixed point. In particular the dynamics of $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) is "wandering" i.e. it does not have any non-trivial recurrence. Also these diffeomorphisms possess attractors and do not preserve any volume measure.

Now assume that M is a compact manifold endowed with a finite volume measure μ (i.e. μ is obtained through a volume form). In the case of homeomorphisms, a volume measure means a measure topologically conjugate to the measure associated to a volume form. These measures are characterized by a theorem due to Oxtobi and Ulam which asserts that a Borel measure μ is topologically conjugate to a volume measure if and only if it has no atoms and is positive on open sets (cf. [O-U]). Furthermore let $\mathrm{Homeo}_\mu(M)$ denote the group of homeomorphisms of M preserving μ (in other words a homeomorphism $h : M \to M$ belongs to $\mathrm{Homeo}_\mu(M)$ if and only if $\mu(\mathcal{B}) = \mu(h(\mathcal{B}))$ for every Borel set $\mathcal{B} \subseteq M$). Also consider the (abstract) free group $F(a_1, \ldots, a_r)$ generated by the symbols $a_1, \ldots, a_r$. Similarly we define the group $\mathrm{Diff}^1_\mu(S)$ of C^1 diffeomorphisms of M preserving μ (note that we also use the word "conservative" to refer to the tranformations which preserve μ).

It is easy to see that there cannot exist a faithful representation ρ from $F(a_1, \ldots, a_r)$ ($r \geq 2$) to $\mathrm{Diff}^1_\mu(S)$ which is structurally stable: note that Poincaré's Recurrence Lemma implies that the non-wandering set of each $\rho(a_i)$

($i = 1, \ldots, r$) coincides with the entire M. Thus the conservative version of Pugh's C^1-Closing Lemma (cf. [P-R]) allows us to approximate the original action (induced by ρ) by an action such that all the $\rho(a_i)$'s ($i = 1, \ldots, r$) share a common periodic point. On the other hand, the usual transversality arguments show that the original action may also be approximated by actions for which the stabilizer of every point is either trivial or infinite cyclic. It follows the non-existence of the structurally stable representation in question.

However there are more subtle questions about the "generic" dynamics of these groups. A classical problem in Ergodic Theory is to decide whether or not ergodic diffeomorphisms (resp. homeomorphisms) are generic in $\mathrm{Diff}^r_\mu(M)$ (resp. $\mathrm{Homeo}_\mu(M)$). At this level of generality, very little is known about this question. One of the by-products of KAM theory (Rüssmann's theorem [Ru]) asserts that ergodic diffeomorphisms of a surface are not generic in class C^r, $r \geq 4$. On the other hand, a deep theorem due to Oxtobi and Ulam establishes that, in an arbitrary compact manifold, ergodic homeomorphisms are generic in class C^0. For non-Abelian free subgroups, we can obtain a slight improvement of the latter theorem namely

Theorem A. *Assume that the dimension of M is different from 4. There exists a residual (i.e. dense G_δ) set $\mathcal{U} \subset \mathrm{Homeo}_\mu(M) \times \mathrm{Homeo}_\mu(M)$ such that, if (f_1, f_2) is in $\mathcal{U}$, then the action of $F(a_1, a_2)$ defined by $a_i \mapsto f_i$ ($i = 1, 2,$) possesses the following properties:*

a) *It is minimal (i.e. the orbit of any point $p \in M$ is dense).*

b) *μ is the unique measure simultaneously preserved by f_1, f_2.*

Remark. The assumption $\dim M \neq 4$ is required by the "Approximation Theorem" (cf. Section 3), but this is an inessential difficulty: it is actually possible to verify the statement even for 4-dimensional manifolds (see comments in [M-P-V] or cf. [O-U]).

The case of C^r ($r \geq 1$) diffeomorphisms is much harder. Recall that KAM theory implies, in particular, that conservative ergodic diffeomorphisms of a surface are not dense for $r \geq 3$. This is, indeed, a consequence of the existence of invariant curves for the twist map. However, in the case of non-Abelian actions, the preceding construction does not allow us to conclude that ergodic diffeomorphisms are not dense since the mentioned curves are not simultaneously invariant by a pair of generic diffeomorphisms. On the other hand, the "topological analogue" of ergodicity, namely the existence of dense orbits, can be verified for surfaces. It is also possible to establish the existence of an ergodic component with positive measure. More precisely, using a result due to Newhouse, we shall prove the following theorem:

Theorem B. *Let S be a compact surface endowed with a C^1-area measure μ. Denote by $\mathrm{Diff}^1_\mu(S)$ the group of C^1-diffeomorphisms of S which preserve μ. Assume that we are given three diffeomorphisms $f_1, f_2, f_3 \in \mathrm{Diff}^1_\mu(S)$ such that at least one among them is not Anosov. Then, arbitrarily C^1-close to f_1, f_2, f_3, there exist diffeomorphisms $\hat{f}_1, \hat{f}_2, \hat{f}_3 \in \mathrm{Diff}^1_\mu(S)$ which generate a group $\widehat{G} \subset \mathrm{Diff}^1_\mu(S)$ having the following property: there exists an open dense set $V \subset S$ invariant under $\widehat{G}$ and such that the action of $\widehat{G}$ restricted to V is minimal and ergodic (with respect the normalized measure on V).*

Of course the main defficiency of Theorem B above is the fact that we do not know whether or not one can always take $V = S$. Actually it is not even clear if $\mu(V) = \mu(S)$.

Since Anosov diffeomorphisms have dense orbits, we deduce Corollary C.

Corollary C. *There exists a residual set $\mathcal{U} \subset \mathrm{Diff}^1_\mu(S) \times \mathrm{Diff}^1_\mu(S)$ such that, if (f_1, f_2) belongs to $\mathcal{U}$, then the action of $F(a_1, a_2)$ defined by sending $a_i \mapsto f_i$ ($i = 1, 2$) has a dense orbit.*[1]

On the other hand, one can ask if the property of having a dense orbit is "generic" to non-Abelian free actions whether or not they preserve a volume. This is however not the case as shown by the example of the Schottky group discussed above. In fact, since the action of a Kleinian group on the complement of its limit set is properly discontinuous, this action has no dense orbit. Furthermore the fact that this action is structural stable implies that no dense orbit can be produced by a perturbation of the generators.

As mentioned in the Introduction, the second part of this paper is devoted to actions admitting a finite set of generators which are close to the identity. Here we leave the low-regularity setting required by Theorems A and B and work in the world of real analytic diffeomorphisms. For dimensions greater than one the main available tool to study these actions is, to the best of my knowledge, Proposition (2.1) below which was obtained with F. Loray in [L-R] (Proposition 4.6 of [L-R]).

[1] added in the proofs: recently Bonatti and Crovisier [B-C] have proved the existence of a dense orbit for a single generic conservative diffeomorphism of a manifold; their methods however do not yield the existence of an ergodic component with positive measure.

Denote by $\mathbb{B}^n \subset \mathbb{C}^n$ the unit ball of $\mathbb{C}^n$. Consider a pseudogroup Γ consisting of *holomorphic maps* from open subset of $\mathbb{C}^n$ to $\mathbb{C}^n$ which satisfies the assumptions below.

a) There exists a sequence $\{h_i\} \subset \Gamma$, $h_i \neq id$ for every $i \in \mathbb{N}$, whose elements are defined on the entire $\mathbb{B}^n$ and, in addition, converge uniformly to the identity on $\mathbb{B}^n$.

b) Γ contains a homothety $F : \mathbb{B}^n \to \mathbb{B}^n$ which is given by $F(z_1, \dots, z_n) = (\lambda_1 z_1, \dots, \lambda_n z_n)$ where $\lambda_i \in \mathbb{R}$ and $0 <| \lambda_1 |\leq \cdots \leq| \lambda_n |< 1$.

Proposition 2.1. (**[L-R]**) *Assume we are given Γ as above. Then there exists $\varepsilon > 0$ and a non-trivial real analytic vector field X defined on $\mathbb{B}(\varepsilon)$ (the ball of radius ε and center at the origin) which possesses the following property: given a relatively compact open set $V \subset \mathbb{B}(\varepsilon)$ and $t_0 \in \mathbb{R}_+$ such that Φ_X^t is defined on V whenever $0 \leq t \leq t_0$, the map $\Phi_X^{t_0} : V \to \Phi_X^{t_0}(V)$ is uniformly approximated on V by elements in Γ.*

Note that the vector field X above can, in fact, be defined on the whole $\mathbb{B}^n$ thanks to the existence of a homothety in the pseudogroup Γ. We also point out that, if $V \subset \mathbb{B}(\varepsilon)$ and $t_0 > 0$ are as in the statement, then the local flow $\Phi_X^t : V \to \Phi_X^t(V)$ is uniformly approximated on V by elements in Γ for every $0 \leq t < t_0$. We shall say that a vector field X possessing the property stated in Proposition (2.1) is *in the closure of* Γ relative to V. In practice the pseudogroup Γ is generated by the restrictions of elements in G to an open set of M. This proposition becomes very effective when the group G contains a Morse-Smale (or gradient-like) diffeomorphism. In particular, denoting by $\mathcal{V} \subset \text{Diff}^\omega(M) \times \text{Diff}^\omega(M)$ the set of the pairs f_1, f_2 in $\text{Diff}^\omega(M)$ such that the group generated by f_1, f_2 contains a Morse-Smale diffeomorphism, we have:

Theorem D. *Let M be an analytic manifold and consider the group* $\text{Diff}^\omega(M)$ *consisting of the real analytic diffeomorphisms of M equipped with its natural analytic topology (cf. Section 4). There exists a neighborhood $\mathcal{U}$ of the identity in* $\text{Diff}^\omega(M)$ *and a residual (i.e. dense G_δ) set $\mathcal{K}$ of $\mathcal{V} \cap (\mathcal{U} \times \mathcal{U})$ such that, if (f_1, f_2) belongs to $\mathcal{K}$, then the dynamics associated to the group G generated by f_1, f_2 has the following properties:*

1. *it is minimal (i.e. all orbits are dense);*

2. *it is ergodic (i.e. every borelian invariant under G has zero or total Lebesgue measure);*

3. *it is topologically rigid (i.e. if G' is generated by $(f_1', f_2') \in \mathcal{K}$ is conjugated to G by a homeomorphism $h : M \to M$ then h is, in fact, an element of* $\mathrm{Diff}^{\omega}(M)$*).*

Perturbations of analytic diffeomorphisms and natural topologies on their group will be discussed in Section 5 so as to make sense of the notion of "generic" action. We should also point out that a very similar but a slightly stronger result was independently obtained by M. Belliart in [Be] by elaborating on the proof of Proposition (2.1) given in [L-R]. The present paper is indeed a revised version of my 2001-preprint [Reb3] which has an overlap with [Be]. However the "generic character" of actions as above, which constitutes the point of view of this article, is not developed in [Be] (for instance, it is not proved in [Be] that a group generated by randomly chosen diffeomorphisms is free). An extra reason to include Theorem D in our dicussion is the fact that it can immediately be derived from a short and self-contained proof of Proposition (2.1) which is provided in Section 6. This proof is already contained in the long discussion of [L-R] but here we give a clearer presentation which makes it promptly accessible to the reader. Compared to [Be], the present proof has two advantages. First it is rather simpler and more "down-to-earth" than the treatment of [Be]. Secondly it is necessary to derive Theorem E below which cannot be obtained with the arguments of [Be]. Actually the main virtue of the version of Proposition (2.1) given here is to single out the essential difficulty to generalize this type of result to other pseudogroups (which might be overlooked in [L-R]). It is the precise understanding of this difficulty that allows us to derive Theorem E as a further application.

To state Theorem E, we consider the group of cocyles (or skew-products) $\mathcal{G}$ (resp. $\mathcal{G}_{\mathbb{C}}$) acting in $\mathbb{S}^1 \times \mathbb{S}^1$ (resp. $\mathbb{S}^2 \times \mathbb{S}^1$) which is constituted by the diffeomorphisms of the form

$$F(x, y) = (Ax, y + u(x)),$$

where A represents the action of an element of $\mathrm{PSL}(2, \mathbb{R})$ (resp. $\mathrm{PSL}(2, \mathbb{C})$) on $\mathbb{S}^1$ (resp. $\mathbb{S}^2$) and u is an analytic function on $\mathbb{S}^1$ (resp. $\mathbb{S}^2$). This class of diffeomorphisms constitutes a classical and interesting object of study in Ergodic Theory.

Given a subgroup G of $\mathcal{G}$, denote by G_{π} the subgroup of $\mathrm{PSL}(2, \mathbb{R})$ (resp. $\mathrm{PSL}(2, \mathbb{C})$) corresponding to the natural projection onto the first component of the elements in G.

Theorem E. *Assume that $G \subset \mathcal{G}$ (or $\mathcal{G}_{\mathbb{C}}$) is not Solvable. Assume also that $G_\pi \subset \mathrm{PSL}(2,\mathbb{R})$ (resp. $\mathrm{PSL}(2,\mathbb{C})$) is not discrete. Then G is ergodic and has all orbits dense in $\mathbb{S}^1 \times \mathbb{S}^1$ (resp. $\mathbb{S}^2 \times \mathbb{S}^1$).*

From Theorem E we can derive additional dynamical properties of groups as above. Particularly interesting is a topological/measurable rigidity phenomenon whose proof would follow the lines of [Reb2] (cf. Corollary F below). Such rigidity contrasts with Furstenberg's celebrated example of a skew-product of $\mathbb{S}^1 \times \mathbb{S}^1$ which is measurably conjugate to a translation but has all orbits denses (so that, in particular, it is not topologically conjugate to a translation, cf. [Ma]). It is surprising that these rigidity statements seem to have been missed in the vast literature about skew-products. Whereas we shall not provide a detailed proof of Corollary F here, since this would force us to do a long detour from the goals of this article, we give a precise (and simplified) statement. In any case the proof will follow from the combination of Theorem E with the technique introduced in Sections 4 of [Reb2]. Several additional details on this kind of argument are provided in the appendix.

Corollary F. *Assume that $G_1, G_2 \subset \mathcal{G}$ are as above. Then the following are equivalent:*

1. *G_1, G_2 are differentiably conjugate.*
2. *G_1, G_2 are topologically conjugate.*
3. *G_1, G_2 are measurably conjugate.*

3 Proof of Theorem A

The proof of Theorem A presented here naturally depends on different aspects of Oxtobi-Ulam work [O-U], nonetheless it was inspired by ideas of S. Alpern which appear in [M-P-V]. Precisely the proof of Proposition (3.1) is an easy adaptation of the argument employed in the proof of the "Approximation Theorem" in [M-P-V], lecture 18.

Let $I^k \subset \mathbb{R}^k$ be the unit cube spanned by the vectors $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ and denote by $S^n I^k$ the $4-$adic subdivision of I^k with order n. This means the following. By definition, $S^1 I^k$ is obtained by dividing each edge of I^k into 4 parts of same length and then forming the obvious k-dimensional cubes with the resulting segments. More generally, $S^{n+1} I^k$ is obtained from $S^n I^k$ by dividing the edges of each cube belonging to $S^n I^k$ into 4 parts of same length and then proceeding as before. It is clear that $S^n I^k$ consists of 4^{nk} cubes $I^k_{n,j}$ ($j = 1, \dots, 4^{nk}$) whose union is the whole I^k. Clearly the length of the edges of $I^k_{n,j}$ is 4^{-n}. Fixed a cube $I^k_{n,j}$ and $\alpha \in \mathbb{R}$, $\alpha > 1$, let $I^k_{n,j,\alpha}$ be the open cube, concentric to $I^k_{n,j}$, but having edges of length $4^{-n}/\alpha$.

For each $s \in \mathbb{N}$ sufficiently large, we choose a finite "covering" $\mathcal{B}_s$ of M consisting of coordinate neighborhoods $\{(B_i, \phi_i)\}$ ($i = 1, \dots, s$) which satisfies the following conditions:

1. $\phi_i(B_i) = I^k \subset \mathbb{R}^k$ where k is the dimension of M;
2. each B_i is strictly contained in some B'_i so that $\phi_i(B'_i)$ is (defined and) a neighborhood of I^k in $\mathbb{R}^k$;
3. the direct image of the restriction of μ to B'_i by ϕ_i is the Lebesgue measure on $\phi_i(B'_i)$;
4. $\mu(B_1) = \mu(B_2) = \cdots = \mu(B_s) = 1/s$;
5. $\bigcup_{i=1}^s \overline{B}_i = M$ where $\overline{B}_i$ stands for the topological closure of B_i;
6. $B_i \cap B_j = \partial B_i \cap \partial B_j$ for $i \neq j$ ($\partial B_i = \overline{B}_i \setminus B_i$).

Note that the collection of the neighborhoods B_i's does not constitute a covering of M since it misses the union of the boundaries of the B_i's. The union of these boundaries can, however, be thougth of as a finite union of hypersurfaces so that it will play no role in our discussion. In view of this we shall refer to the union of the B_i's as forming a "covering" (using quotes) for M.

Now using the "covering" above, we are able to define the 4−adic subdivisions of M as follows. The 4−adic subdivision of M with order n, $S^n M$, consists of the cells ("cubes") $\phi_i^{-1}(I^k_{n,j})$ ($i = 1, \dots, l_s$; $j = 1, \dots, 4^{nk}$) where $I^k_{n,j}$ are the cells (cubes) of the 4−adic subdivision of I^k with order n (again this procedure misses the union of the boundaries ∂B_i of B_i, however this will be irrelevant to our discussion). Similarly we define the sets $\phi_i^{-1}(I^k_{n,j,\alpha})$.

On the other hand, fixed s and the covering $\mathcal{B}_s$, consider the set $\mathcal{U}_s \subset \mathrm{Homeo}_\mu(M) \times \mathrm{Homeo}_\mu(M) = (\mathrm{Homeo}_\mu(M))^2$ formed by the pairs (f_1, f_2) such that, for any point $p \in M$, the G-orbit of p intersects all the B_i's ($i = 1, \dots, l_s$; where G stands for the group generated by f_1, f_2).

The set $\mathcal{U}_s$ is clearly open for the C^0−topology. Moreover one has:

Proposition 3.1. *Fixed $s \in \mathbb{N}^*$ and $\mathcal{B}_s$, the subset $\mathcal{U}_s \subset (\mathrm{Homeo}_\mu(M))^2$ is dense.*

Consider the set $\mathcal{R}^n M = M \setminus \bigcup_{i,j} \phi_i^{-1}(I^k_{n,j})$ (recall that the sets $I^k_{n,j}$ are open by definition). In other words, $\mathcal{R}^n M$ consists of the union of the boundaries of the B_i's and the union of the pre-images by the corresponding ϕ_i's of the complements $I^k \setminus \bigcup_j I^k_{n,j}$. Lemma (3.2) below is very elementary.

Lemma 3.2. *Assume that $n_0 \in \mathbb{N}^*$ is fixed. The set of homeomorphisms $h \in$ $\mathrm{Homeo}_\mu(M)$ whose dynamics admits no orbit entirely contained in $\mathcal{R}^{n_0} M$ is open and dense.*

Proof. Note that h has no orbit contained in $\mathcal{R}^{n_0} M$ if and only if the orbit of every point $p \in \mathcal{R}^{n_0} M$ intersects $M \setminus \mathcal{R}^{n_0} M$. Since $\mathcal{R}^{n_0} M$ can be thougth of as a finite union of "smooth hypersurfaces" (modulo using good coordinates as ϕ_i) the statement follows at once. $\square$

The next result is the main ingredient of the proof of Proposition (3.1). As mentioned the Approximation Theorem of [M-P-V], lecture 18 is stated for $\alpha = \sqrt[k]{2}$ but a quick look at the proof makes it clear that it works also for any $\alpha > 1$.

Theorem 3.3. (**Alpern in [M-P-V], lecture 18**). *Assume we are given $h \in$ $\mathrm{Homeo}_\mu(M)$, $\alpha \in \mathbb{R}$ satisfying $\alpha < 1$ and $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ depending only on ε (and not on α) and $\hat{h} \in \mathrm{Homeo}_\mu(M)$, ε-close to h, which cyclically permutes the cubes (cells) $\phi_i^{-1}(I^k_{n_0,j,\alpha})$ of $S^{n_0} M$ ($i = 1, \ldots, l_s$; $j = 1, \ldots, 4^{nk}$). In other words, chosen n_0 large enough, we can find $\hat{h}$ ε-close to h and satisfying the desired condition for any $\alpha > 1$.* $\square$

Proof of Proposition (3.1). Recall that $\mathcal{B}_s$ is fixed and assume we are given a pair of homeomorphisms (f_1, f_2) in $(\mathrm{Homeo}_\mu(M))^2$. Given $\varepsilon > 0$, we have to check the existence of $(\hat{f}_1, \hat{f}_2) \in \mathcal{U}_s \subset (\mathrm{Homeo}_\mu(M))^2$ satisfying $\| \hat{f}_1 - f_1 \| < \varepsilon$ (resp. $\| \hat{f}_2 - f_2 \| < \varepsilon$).

According to Lemma (3.2), we can perturb f_2 into $\hat{f}_2$ so that $\mathcal{R}^{n_0} M$ contains no non-trivial minimal set of $\hat{f}_2$. In other words, the $\hat{f}_2$-orbit of every point $q \in \mathcal{R}^{n_0}$ intersects the complement of $\mathcal{R}^{n_0}$. Thus there is an open neighborhood $V \subset M$ of $\mathcal{R}^{n_0}$ with the same property, namely we can associate to each point q of V an iterate $\hat{f}_2^{m_q}$ of $\hat{f}_2$ so that $\hat{f}_2^{m_q}(q)$ lies on $M \setminus V$.

Finally we take $\alpha < 1$ so that the union $V \cup \bigcup \phi_i^{-1}(I^k_{n_0,j,\alpha})$ covers the whole of M (where $\phi_i^{-1}(I^k_{n_0,j,\alpha})$ are contained in the cells of $S^{n_0} M$). Using Theorem (3.3) we obtain $\hat{f}_1$ cyclically permuting the cubes $\phi_i^{-1}(I^k_{n_0,j,\alpha})$ and ε-close to f_1.

To finish the proof it is enough to verify that $(\hat{f}_1, \hat{f}_2)$ as above belongs to $\mathcal{U}_s$. This is however obvious: if $p \in M$ belongs to $\bigcup \phi_i^{-1}(I^k_{n_0,j,\alpha})$ then its $\hat{f}_1$-orbit intersects all the sets in $\mathcal{B}_s$; if $p \in M$ belongs to V, then there is a point of its $\hat{f}_2$-orbit lying in $M \setminus V$ and therefore its G-orbit intersects all the sets in $\mathcal{B}_s$ (where G stands for the group generated by $\hat{f}_1, \hat{f}_2$). The proposition is proved. $\square$

Notice that Proposition (3.1) enables us to prove the genericity of groups $G \subset \mathrm{Homeo}_\mu(M)$ as in the statement of Theorem A which have minimal dynamics. In fact, it is enough to consider a sequence of coverings $\mathcal{B}_1, \mathcal{B}_2, \dots,$ satisfying the conditions in the beginning of this section, and such that the diameters of the open sets in $\mathcal{B}_s$ converge uniformly to *zero*. For such a sequence let $\mathcal{U}_1, \mathcal{U}_2, \dots$ be the corresponding open dense sets given by Proposition (3.1). Clearly any element in the intersection $\bigcup_{s=1}^{\infty} \mathcal{U}_s$ is minimal and, on the other hand, Baire's theorem asserts that this intersection is "generic".

To complete the proof of Theorem A, it remains to analyse the structure of invariant measures for generic subgroups of $\mathrm{Homeo}_\mu(M)$. For the rest of this section we suppose fixed the covering $\mathcal{B}_s$ (which was defined at the beginning of the section). We begin with the lemma below which is still an elementary generalization of Lemma (3.2).

Given $n_0 \in \mathbb{N}$ and $\varepsilon > 0$, denote by $\mathcal{W}_\varepsilon^{n_0}$ the subset of $\mathrm{Homeo}_\mu(M)$ consisting of those homeomorphisms f satisfying the following condition: if ν is a Borel probability invariant under f then $\nu(\mathcal{R}^{n_0}) < \varepsilon$.

Lemma 3.4. *Assume $n_0 \in \mathbb{N}$ is fixed. Given $\varepsilon > 0$, the set $\mathcal{W}_\varepsilon^{n_0}$ is dense in* $\mathrm{Homeo}_\mu(M)$.

Proof. Let us prove that the complement of $\mathcal{W}_\varepsilon^{n_0}$ is closed and has empty interior. Assume that $h \in \mathrm{Homeo}_\mu(M)$ has an invariant probability measure satisfying $\nu(\mathcal{R}^{n_0}) \geq \varepsilon$. Notice that, in this case, $\mathcal{R}^{n_0}$ cannot have more than $1/\varepsilon$ disjoint images under h. In other words, if $m > 1/\varepsilon$, then for some $i \in \{1, \dots, m\}$, $h^i(\mathcal{R}^{n_0}) \cap \mathcal{R}^{n_0} \neq \emptyset$. Therefore the idea is to use an argument of general position to show that f can be approximated by a homeomorphism for which $\mathcal{R}^{n_0}$ has "many disjoint" images. Let us make this idea more precise.

First suppose that M is a surface so that $\mathcal{R}^{n_0}$ consists of a union of lines. Choose $m > 1/\varepsilon$. By a "transversality argument", it is possible to aproximate h by h_1 so that the $h_1^i(\mathcal{R}^{n_0}) \cap h_1^j(\mathcal{R}^{n_0})$ is reduced to a finite number of points for every $i, j \in \{1, \dots, m\}$, $i \neq j$. Now we can approximate h_1 by h_2 so that

$$h_1^i(\mathcal{R}^{n_0}) \cap h_1^j(\mathcal{R}^{n_0}) = h_2^i(\mathcal{R}^{n_0}) \cap h_2^j(\mathcal{R}^{n_0})$$

and, furthermore, the h_2-orbit of any point in $h_2^i(\mathcal{R}^{n_0}) \cap h_2^j(\mathcal{R}^{n_0})$, $i \neq j$ is infinite. Thus, if ν_2 is a probability preserved by h_2, one has $\nu_2(h_2^i(\mathcal{R}^{n_0}) \cap h_2^j(\mathcal{R}^{n_0})) = 0$ whenever $i \neq j$. Hence

$$\nu_2 \left(\bigcup_{i=1}^{m} h_2^i(\mathcal{R}^{n_0}) \right) = m \,.\, \nu_2 \left(\mathcal{R}^{n_0}\right) \leq 1 \,.$$

We conclude that $\nu_2(\mathcal{R}^{n_0}) \leq 1/m < \varepsilon$.

For a general manifold M we proceed in a recurrent way on the intersections $h^i(\mathcal{R}^{n_0}) \cap h^j(\mathcal{R}^{n_0})$. Precisely, by the transversality argument, the dimension of these intersections will be smaller than $\dim(M) - 1$ (where $\dim(M)$ stands for the dimension of M). So we can approximate h by h_1 so that

$$\nu_1(h_1^i(\mathcal{R}^{n_0}) \cap h_1^j(\mathcal{R}^{n_0})) < \delta \,,$$

for every probability preserved by h_1 and any $\delta > 0$. If δ is small enough, we then conclude that

$$m \,.\, \nu_1((\mathcal{R}^{n_0})) \leq 1 - m(m-1)\delta < \varepsilon \,.$$

The proof of the lemma is over. $\square$

Remark 3.5. Assume that h belongs to $\mathcal{W}_\varepsilon^{n_0} \subset \mathrm{Homeo}_\mu(M)$. We claim the existence of a relatively compact open neighborhood V of $\mathcal{R}^{n_0}M$ such that $\nu(V) < 2\varepsilon$ for every Borel probability ν preserved by h. In fact suppose for a contradiction the claim is false. Thus there is a sequence of relatively compact neighborhoods V_i and a sequence of measures ν_i preserved by h which satisfy the conditions below:

- $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots \supseteq V_i \supseteq \cdots$;
- $\bigcap_{i=1}^{\infty} V_i = \mathcal{R}^{n_0}M$;
- $\nu_i(V_i) > 2\varepsilon$ for every i.

Since the space of Borel probabilities is compact (cf. below), we can suppose that ν_i converges to ν which is automatically preserved by h. Since $\nu(\mathcal{R}^{n_0}M) < \varepsilon$, there is a neighborhood U of $\mathcal{R}^{n_0}M$ such that $\nu(U) \leq 3\varepsilon/2$. For sufficiently large i, one has $V_i \subset U$, thus $\nu_i(U) - \nu(U) \geq \varepsilon/2 > 0$ which is a contradiction with the fact that $\nu_i \to \nu$. This proves our claim.

It is well-known that the space of all Borel probabilities on M is a compact metric space (cf. for instance [Ma]). Denote by $d(.)$ the associated metric. To construct $d(.)$, we consider a countable set of open balls $\{\mathbf{B}^i\}_{i\in\mathbb{N}}$ defining the topology of M. If ν_1, ν_2 are probability measures on M, we set

$$d(\nu_1, \nu_2) = \sum_{i=1}^{\infty} \frac{\| \nu_1(\mathbf{B}^i) - \nu_2(\mathbf{B}^i) \|}{2^i}.$$

Consider the set $\mathfrak{W}^n \subset (\mathrm{Homeo}_\mu(M))^2$ of pairs (f_1, f_2) which do not preserve any borelian probability ν whose distance to μ is greater or equal $1/n$.

Lemma 3.6. $\mathfrak{W}^n \subset (\mathrm{Homeo}_\mu(M))^2$ *is an open set.*

Proof. Suppose for a contradiction that the statement is false. Then there is $(f_1, f_2) \in \mathfrak{W}^n$ and a sequence $\{(\hat{f}_{1l}, \hat{f}_{2l})\} \subset (\mathrm{Homeo}_\mu(M))^2 \setminus \mathfrak{W}^n$ such that each coordinate $\{\hat{f}_{il}\}$ converges uniformly to f_i. For each l, we consider a measure ν_l preserved by $(\hat{f}_{1l}, \hat{f}_{2l})$ and having distance to μ at least equal to $1/n$. Clearly $\{\nu_l\}$ can be supposed to converge towards some measure ν which also satisfies $d(\nu, \mu) \geq 1/n$, in particular ν is not preserved by (f_1, f_2).

Because ν is not preserved by (f_1, f_2), there is an element $\mathbf{f}$ in the group $< f_1, f_2 >$ generated by f_1, f_2 and an open set $U \subset M$ such that $\nu(U) < \nu(\mathbf{f}(U))$. Next consider the elements $\hat{\mathbf{f}}_l \in < \hat{f}_{1l}, \hat{f}_{2l} >$ corresponding to $\mathbf{f}$ in the obvious way (in particular $\{\hat{\mathbf{f}}_l\} \to \mathbf{f}$ uniformly). In addition note that we can find a relatively compact set $V \subset \mathbf{f}(U)$ such that $\nu(V) > \nu(U)$. However, since $\{\hat{\mathbf{f}}_l\} \to \mathbf{f}$, for l large enough V is enclosed in $\hat{\mathbf{f}}_l(U)$ so that

$$\liminf_{l\to\infty} \nu_l(\hat{\mathbf{f}}_l(U)) \geq \liminf_{l\to\infty} \nu_l(V) = \nu(V).$$

On the other hand $\nu_l(U)$ converges to $\nu(U)$ which is strictly less than $\nu(V)$, so we finally obtain $\nu_l(U) < \nu(V) \leq \nu_l(\hat{\mathbf{f}}_l(U))$ for l large enough. The resulting contradiction establishes the lemma. □

We are ready to complete the proof of Theorem A.

Proof of Theorem A. Since we have seen that there is a residual subset of $(\mathrm{Homeo}_\mu(M))^2$ whose elements have all orbits dense, we just need to verify the existence of another residual subset of $(\mathrm{Homeo}_\mu(M))^2$ whose elements have only μ as common invariant measure. After Lemma (3.6), we just need to check that the sets $\mathfrak{W}^n \subset (\mathrm{Homeo}_\mu(M))^2$ are dense.

Assume we are given $\delta > 0$ and consider the collection $\{\mathbf{B}^1, \mathbf{B}^2, \ldots, \mathbf{B}^m\}$ of the balls involved in the definition of the metric $d(.)$. In order to prove that $\mathfrak{W}^n$ is dense, we need to prove that a pair $(f_1, f_2) \in (\mathrm{Homeo}_\mu(M))^2$ can be approximated by another pair $(\hat{f}_1, \hat{f}_2) \in (\mathrm{Homeo}_\mu(M))^2$ with the following property: if ν is a borelian probability preserved by $\hat{f}_1, \hat{f}_2$, then

$$|| \nu(\mathbf{B}^i) - \mu(\mathbf{B}^{\mathbf{i}}) || < \delta \text{ for every } i = 1, \dots, m .$$

Indeed, if $\hat{f}_1, \hat{f}_2$ are as above for appropriate δ and m, it follows from the definition of $d(.)$ that the pair $(\hat{f}_1, \hat{f}_2)$ belongs to $\mathfrak{W}^n$.

Thus assume from now on that $\delta > 0$ and $m \in \mathbb{N}$ are fixed. Given $\varepsilon > 0$, we need to find $\hat{f}_1$ (resp. $\hat{f}_2$) ε-close to f_1 (resp. f_2) satisfying the above condition for every $\mathbf{B}^i$ $i = 1, \dots, m$. Using the notation of Theorem (3.3), choose n_0 so large that Theorem (3.3) does apply with respect to ε and

$$\frac{K_2^i - K_1^i}{\sharp\, S^{n_0} M} < \frac{\delta}{10} ,$$

for every $i = 1, \dots, m$, where:

1. $\sharp\, S^{n_0} M$ denotes the number of cells in $S^{n_0} M$ (i.e. the cardinality of $S^{n_0} M$ which is equal to $l_s 4^{n_0 k}$);

2. K_1^i $(i = 1, \dots, m)$ is the maximum number of cells σ_i of $S^{n_0} M$ whose union is contained in $\mathbf{B}^i$;

3. K_2^i $(i = 1, \dots, m)$ is the minimum number of cells σ_i of $S^{n_0} M$ whose union contains $\mathbf{B}^i \setminus \mathcal{R}^{n_0} M$ (by a small abuse of notation, in this case we simply say that the union of the σ_i's *contains* $\mathbf{B}^i$).

Using Lemma (3.4) we approximate f_2 by $\hat{f}_2$ belonging to $\mathcal{W}^{n_0}_{\delta/20}$. Next let V be an open neighborhood of $\mathcal{R}^{n_0} M$ satisfying the following condition:

$(*)$ – if ν is a Borel probability invariant under $\hat{f}_2$, then $\nu(V) < \delta/10$.

The existence of V is ensured by Remark (3.5). Now we take $\alpha > 1$ so close to 1 that $V \cup \bigcup \phi_i^{-1}(I^k_{n_0,j,\alpha})$ covers the whole of M. Finally we approximate f_1 by $\hat{f}_1$ so that $\hat{f}_1$ permutes the cubes $\phi_i^{-1}(I^k_{n_0,j,\alpha})$ as in Theorem (3.3).

To finish the proof, it is enough to verify that $(\hat{f}_1, \hat{f}_2)$ satisfy the above condition on δ and $\mathbf{B}^i$. Thus consider a probability ν simultaneously invariant under $\hat{f}_1, \hat{f}_2$.

Note that all the cubes $\phi_i^{-1}(I^k_{n_0,j,\alpha})$ have the same ν-measure since they are permuted by $\hat{f}_1$. Furthermore the estimate below does hold for the measure of the union of all these cubes:

$$\nu\left(\bigcup_{S^{n_0} M} \phi_i^{-1}(I^k_{n_0,j,\alpha})\right) < 1 - \nu(V) < 1 - \delta/10 .$$

Denote by σ^α one of these cubes. Because they are also pairwise disjoint, one obtains

$$\frac{1}{\sharp\, S^{n_0} M} \geq \nu\,(\sigma^\alpha) \geq \frac{1-\delta/10}{\sharp\, S^{n_0} M}\,.$$

It follows that

$$\frac{K_2^i}{\sharp\, S^{n_0} M} \geq \nu\,(\mathbf{B}^i) \geq \frac{K-1^i(1-\delta/10)}{\sharp\, S^{n_0} M}\,.$$

On the other hand

$$\frac{K_2^i}{\sharp\, S^{n_0} M} \geq \mu(\mathbf{B}^i) \geq \frac{K_1^i}{\sharp\, S^{n_0} M}\,.$$

Thus

$$||\ \mu(\mathbf{B}^i) - \nu(\mathbf{B}^i)\ || \leq \frac{K_2^i - K_1^i}{\sharp\, S^{n_0} M} + \frac{\delta K_1^i}{10\sharp\, S^{n_0} M} \leq \frac{\delta}{10} + \frac{\delta}{10} < \delta\,.$$

The proof of the theorem is completed. □

4 Proof of Theorem B

This section is devoted to the proof of Theorem B. Thus S denotes a compact surface, μ is a C^1-area measure and f_1, f_2 are elements of $\mathrm{Diff}^1_\mu(S)$. Without loss of generality, we can suppose that f_1 is not of Anosov type.

Recall that a fixed point p of a diffeomorphism f of S is called *elliptic* if the eigenvalues of f at p are non-real numbers of modulus 1. More generally, if p is periodic for f with period k, then p is said elliptic for f if the eigenvalues of f^k at p are non-real and have modulus 1.

The non-wandering set of a conservative diffeomorphism coincides with the whole ambient manifold by virtue of Poincaré Recurrence Lemma. Given a point $p \in S$ and a diffeomorphism $f \in \mathrm{Diff}^1_\mu(S)$, the Closing Lemma (conservative version, [P-R]) then states that f can be C^1-approximated by a diffeomorphism $\tilde{f} \in \mathrm{Diff}^1_\mu(S)$ for which p is a periodic point. Furthermore, unless f is of Anosov type, a theorem due to Newhouse [Ne] asserts that we can, in fact, assume that p is elliptic for $\tilde{f}$.

Definition 4.1. *A periodic point p of period k for $h \in \mathrm{Diff}^1_\mu(S)$ will be called a local rotation ("Siegel disk") if and only if there is a relatively compact neighborhood U of p satisfying the two conditions below:*

1. *The boundary ∂U of U is invariant under h^k.*

2. *There is a local coordinate ϕ taking the closure of U, $\overline{U}$, to the closed unit ball $\overline{\mathbb{B}}$ of $\mathbb{C}$ in which the restriction of h^k to U has the form*

$$h^k(z) = \exp(i\theta).z\,, \quad \textit{where } \theta/2\pi \textit{ is irrational.}$$

The proof of our first lemma relies heavily on the theorem of Newhouse mentioned above.

Lemma 4.2. *Assume that $h \in \mathrm{Diff}^1_\mu(S)$ is not of Anosov type. Then, given $\varepsilon > 0$ and a point $p \in S$, there is a homeomorphism $\hat{h} \in \mathrm{Diff}^1_\mu(S)$ having C^1-distance to h less than ε for which p is a local rotation.*

Proof. First we approximate h by $\hat{h}_1 \in \mathrm{Diff}^1_\mu(S)$ for which p is a periodic elliptic point with period $k \in \mathbb{N}^*$. Next let $B_\delta(p)$ be the ball of radius δ centered at p. Choose $\delta > 0$ so that the sets $B_\delta(p), \hat{h}_1(B_\delta(p)), \ldots, \hat{h}_1^{k-1}(B_\delta(p))$ are pairwise disjoint. Now observe that the boundary of $\hat{h}_1^k(B_\delta(p))$, $\partial\hat{h}_1^k(B_\delta(p))$, converges in the C^1 topology to $\partial B_\delta(p)$ (the boundary of $B_\delta(p)$) when δ goes to *zero* even after a dilation of ratio $1/\delta$ of $B_\delta(p)$. Choosing δ very small, it follows that $\hat{h}_1$ can be approximated by $\hat{h}_2$ so that $\hat{h}_2^k$ leaves $\partial B_\delta(p)$ invariant i.e. $\hat{h}_2^k(\partial B_\delta(p)) = \partial B_\delta(p)$.

Since the mapping from $\partial B_\delta(p)$ to $\partial B_\delta(p)$ induced by $\hat{h}_2^k$ is isotopic to the local map given by the derivative of $\hat{h}_2^k$ at p, it follows that $\hat{h}_2^k$ can be C^1-deformed inside $\hat{h}_2^{k-1}(B_\delta(p))$ into $\hat{h}$ so that p becomes a local rotation of period k for $\hat{h}$. This proves the lemma. □

We now introduce the definition of vector fields in the closure of a group following previous works as [Na] and [Reb1].

Definition 4.3. *Let U be an open set on a manifold M equipped with a vector field X. Denote by Φ_X the local flow of X on U and assume that there is a group G acting on M. We say that X is in the C^1-closure (resp. C^k, C^∞-closure) of G if and only if the following holds: given a relatively compact subset $V \subset U$ and $t_0 \in \mathbb{R}$ such that Φ^t_X is defined on V whenever $0 \leq t \leq t_0$, the mapping $\Phi^{t_0}_X : V \to \Phi^{t_0}_X(V)$ is the C^1-limit (resp. C^k, C^∞-limit) on V of the restriction to V of a sequence of elements in G.*

Note that the above definition is natural under the action of G in the sense that, if X defined on U is in the C^1-closure of G and h belongs to G, then h_*X defined on $h(U)$ is in the C^1-closure of G as well. The next lemma contains the main idea in the proof of Theorem B.

Lemma 4.4. *Let $G =< f_1, f_2 >\subset \mathrm{Diff}^1_\mu(S)$ be as before. Assume we are given $\varepsilon > 0$ and a point p in S. Then there exists $\hat{f}_1 \in \mathrm{Diff}^1_\mu(S)$ (resp. $\hat{f}_2 \in \mathrm{Diff}^1_\mu(S)$), which is ε-close to f_1 (resp. f_2) with respect to the C^1-distance, such that the conditions below hold.*

1. *There is a neighborhood U of p in S equipped with 2 vector fields X, Y which are linearly independent at every point of U;*
2. *X, Y are contained in the C^1-closure of $\widehat{G}$, the group generated by $\hat{f}_1, \hat{f}_2$.*

Proof. Using Lemma (4.2), we can suppose that p is a local rotation with period $k_1 \in \mathbb{N}$ for $\hat{f}_1$. Let $\psi : V \subset S \to \overline{D} \subset \mathbb{C}$ be the coordinate in which $\hat{f}_1^{k_1}$ becomes

$$\hat{f}_1^{k_1}(z) = e^{i\theta} z\,, \quad \text{where} \quad \| z \| < 1 \;\text{ and }\; \theta/2\pi \;\text{ is irrational}\,.$$

Consider the vector field $\mathbf{Z}$ defined on $\overline{D}$ and given by $\mathbf{Z}(z) = \| z \| \, (\partial/\partial x - \partial/\partial y)$. Because θ is irrational, the powers of $\hat{f}_1^{k_1}$ approximate the flow of $\mathbf{Z}$. In particular the vector field $Z = \psi^*\mathbf{Z}$ defined on V belongs to the closure of the cyclic group generated by $\hat{f}_1^{k_1}$.

We can suppose the existence of a neighborhood $V_1 \subset V$ of p and a number $k_2 \in \mathbb{N}^*$ such that $U = V_1 \cap \hat{f}_2^{k_2}(V_1)$ is still a neighborhood of p. Indeed, it would be enough to apply the Closing Lemma to f_2. Obviously we can also have $\hat{f}_2^{k_2}(p) \neq p$. Hence the vector field $X = (\hat{f}_2^{k_2})^* Z$ is defined on U and verifies $X(p) \neq 0$. Furthermore X belongs to the closure of the group generated by $\hat{f}_1, \hat{f}_2$. Finally recalling that the derivative of $\hat{f}_1^{k_1}$ at p is an irrational rotation, it results the the vector field $Y = (\hat{f}_1^{k_1})^* X$ is such that $X(p), Y(p)$ are linearly independent at p. Therefore they remain linearly independent in a neighborhood U of p (modulo reducing U). Since they are both contained in the closure of the group generated by $\hat{f}_1, \hat{f}_2$, the proof of the lemma is finished. $\square$

Assume that $G \subset \mathrm{Diff}^1_\mu(S)$ acts on S and let $U \subset S$ be an open set equipped with vector fields X, Y in the closure of G which are in addition linearly independent at every point of U. Observe that the G-orbit of any point in U is dense in U: in fact, if p, q are distinct points of U, then they can be "joined following the flows of X, Y". Because X, Y are in the closure of G, we can find a sequence of elements $\{h_i\} \subset G$ such that $h_i(p)$ converges to q. This shows that these orbits are dense in U. Furthermore a similar argument holds in the sense of "local ergodicity", it suffices to take into account the fact that the sequence $\{h_i\}$ actually converges C^1 on U to the composition of local flows of X, Y.

We are now able to prove the Theorem B.

Proof of Theorem B. Let f_1, f_2 be as in the statement of the theorem in question. We want to verify the existence of $\hat{f}_1, \hat{f}_2$ generating a group $\widehat{G}$ for which it is possible to select an open set $V \subset S$ having the desired properties. Clearly the C^1-distance between f_1 and $\hat{f}_1$ (resp. f_2 and $\hat{f}_2$) must be less than ε, for any $\varepsilon > 0$ previously fixed. We shall construct $\hat{f}_1, \hat{f}_2$ by a step-wise procedure.

Choose $p_1 \in S$. According to Lemma (4.4), we can find $\hat{f}_{1,1}$ (resp. $\hat{f}_{2,1}$) $\varepsilon/2$-close to f_1 (resp. f_2) in the C^1-distance such that the group $\widehat{G}_1$ generated by these diffeomorphisms contains in its closure vector fields X_1, Y_1 defined and linearly independent on a neighborhood U_1 of p_1. Denote by V_1 the orbit of U_1 under $\widehat{G}_1$. In view of the preceding discussion, it follows that any point in V_1 has orbit under $\widehat{G}_1$ everywhere dense in V_1. Therefore, if V_1 is dense in S, the proof of the theorem is over.

Hence we suppose that V_1 is not dense. Consider a point p_2 in $S \setminus \overline{V}_1$ (where $\overline{V}_1$ stands for the topological closure of V_1 in S). Repeating the procedure above, we can obtain $\hat{f}_{1,2}$ (resp. $\hat{f}_{2,2}$) $\varepsilon/4$-close to $\hat{f}_{1,1}$ (resp. $\hat{f}_{2,1}$) in the C^1-distance, which generate a group $\widehat{G}_2$ having the following property: there is a neighborhood U_2 of p_2 equipped with linearly independent vector fields X_2, Y_2 in the closure of $\widehat{G}_2$. Furthermore, since $\overline{V}_1$ is invariant under $\widehat{G}_1$, we can also assume that the restriction of $\hat{f}_{1,2}$ (resp. $\hat{f}_{2,2}$) to $\overline{V}_1$ coincides with the restriction of $\hat{f}_{1,1}$ (resp. $\hat{f}_{2,1}$) to the same set. Finally denote by V_2 the orbit of U_2 under $\widehat{G}_2$.

If V_2 is dense in $S \setminus \overline{V}_1$, we stop our procedure. Otherwise we continue by choosing p_3 in $S \setminus \overline{V}_1 \cup \overline{V}_2$. applying successively this argument we obtain by transfinite induction two diffeomorphisms $\hat{f}_1, \hat{f}_2$ in $\mathrm{Diff}^1_\mu(S)$ generating a group $\widehat{G}$ which satisfies all the conditions below.

1. $\hat{f}_1$ (resp. $\hat{f}_2$) is ε-close to f_1 (resp. f_2) with respect to the C^1-distance.
2. There are a countable (maybe finite) number of open sets $V_1, V_2, \ldots$ invariant under $\tilde{G}$.
3. The union $\bigcup_{i=1}^{\infty} V_i$ is dense in S.
4. The action of $\widehat{G}$ on V_i is minimal (i.e. has all orbits dense) and ergodic with respect to the Lebesgue measure of V_i.

To conclude the proof, we just need to see that all these V_i's can be "connected" by perturbing f_3. The complement of $\bigcup_{i=1}^{\infty} V_i$, namely the union $\bigcup_{i=1}^{\infty} \partial V_i$ of the boundaries of the V_i's cannot be stably invariant under f_3. Actually by perturbing f_3 into $\hat{f}_3$ we can ensure that any set contained in $\bigcup_{i=1}^{\infty} \partial V_i$ and invariant by $\hat{f}_3$

is totally disconnected. Hence the set $\bigcup_{i=1}^{\infty} V_i$ will be minimal for the group generated by $\hat{f}_1$, $\hat{f}_2$, $\hat{f}_3$. The proof of the theorem is over. $\square$

Proof of Corollary C. Just note that, for the purposes of a dense orbit, we can connect the sets V_i's by conveniently perturbing $\hat{f}_1$, $\hat{f}_2$ themselves. $\square$

5 Groups of analytic diffeomorphisms

We now begin the second part of the present work which is devoted to groups of analytic diffeomorphisms. In this section we shall discuss some basic results especially those borrowed from [B-T] and [Gh]. From now to the end of this work M will stand for a compact analytic manifold and $\mathrm{Diff}^{\omega}(M)$ will denote the group of analytic diffeomorphisms of M.

By virtue of a result due to Grauert, there exists a real analytic embedding of M in some Euclidean space $\mathbb{R}^N$. Considering $\mathbb{R}^N$ as contained in $\mathbb{C}^N$, we can define a *complexification* of M which is an open complex manifold $\widetilde{M}$ of complex dimension equal to the real dimension of M. Two complexifications of M coincide on a neighborhood of M and we fix one of them.

It suffices to introduce the analytic topology on the space $C^{\omega}(M, \mathbb{R}^l)$ consisting of the $\mathbb{R}^l$-valued analytic functions on M. Actually, since M is embedded in $\mathbb{R}^N$, the group of analytic diffeomorphisms $\mathrm{Diff}^{\omega}(M)$ of M form a closed subset of $C^{\omega}(M, \mathbb{R}^N)$. Thus this group is naturally endowed with the induced topology.

Given $\tau > 0$, let $\widetilde{M}_\tau$ denote the set of points of $\widetilde{M}$ whose Euclidean distance (i.e. the ordinary distance between points in $\mathbb{C}^N$) to M is less than τ. Next, if $\tau, \varepsilon \in \mathbb{R}_+^*$, we define the set $\mathcal{U}(\tau, \varepsilon) \subset C^{\omega}(M, \mathbb{R}^l)$ by

$$\mathcal{U}(\tau, \varepsilon) = \{h \in C^{\omega}(M, \mathbb{R}^l) \ ; \ \sup_{z \in \widetilde{M}_\tau} || \tilde{h}(z) || < \varepsilon\}$$

where $\tilde{h}$ stands for the holomorphic extension of h to $\widetilde{M}_\tau$ (notation: if h does not admit a holomorphic extension to $\widetilde{M}_\tau$, we set $\sup_{z \in \widetilde{M}_\tau} || \tilde{h}(z) || = +\infty$). To define the analytic topology on $C^{\omega}(M, \mathbb{R}^l)$, we just need to impose the following conditions:

$\imath$) A basis of neighborhoods for the constant map taking M to the origin of $\mathbb{R}^l$ is given by the sets $\mathcal{U}(\tau, \varepsilon)$, $\tau, \varepsilon \in \mathbb{R}_+^*$.

$\imath\imath$) A basis of neighborhoods for a given $h \in C^{\omega}(M, \mathbb{R}^l)$ is obtained by translation of the above mentioned basis.

According to Takens, [Ta], the analytic topology above defined possesses Baire Property.

Before going further and discuss analytic perturbations, we want to recall the notion of pseudo-solvable groups introduced in [Gh]. So consider a finitely generated group G along with a generating set $S = \{f_1, \ldots, f_l\}$ (the reader can think of G as a subgroup of $\mathrm{Diff}^{\omega}(M)$). Following [Gh] we define a sequence $\Sigma(S, n)$ of subsets of G as follows:

$\imath$) $\Sigma(S, 0) = S$

$\imath\imath$) $\Sigma(S, n+1) = \{[g^{\pm 1}, h^{\pm 1}]\}$ where $g \in \Sigma(S, n)$, $h \in \Sigma(S, n-1) \cup \Sigma(S, n)$ ($h \in \Sigma(S, 0)$ if $n = 0$) and $[g, h] = g \circ h \circ g^{-1} \circ h^{-1}$.

Definition 5.1. (**[Gh]**) *The group G is said to be pseudo-solvable if, for some finite generating set S of G, the sequence $\Sigma(S, n)$ degenerates into $\{id\}$ for n large enough.*

The interest of this definition is justified by the theorem below which is due to Ghys.

Theorem 5.2. (**Ghys, [Gh]**) *There exists an open neighborhood $\mathcal{U}$ of the identity in* $\mathrm{Diff}^{\omega}(M)$ *such that, if G is a group which is not pseudo-solvable and admits a finite generating set $S = \{f_1, \ldots, f_l\}$ contained in $\mathcal{U}$, then G contains a sequence of diffeomorphisms $\{h_k\}$ satisfying the following conditions:*

1. *$h_k \neq id$ for all k.*
2. *Each h_k possesses a holomorphic extension $\tilde{h}_k$ to $\widetilde{M}_{\tau}$ for some uniform $\tau > 0$.*
3. $\sup_{z \in \widetilde{M}_{\tau}} \| \tilde{h}_k(z) - z \|$ *goes to zero when k goes to infinity.*

An immediate consequence of item 3 and Cauchy's Formula is that the sequence $\{h_k\}$ converges C^{∞} to the identity on M. In order to apply Ghys's theorem to our problems, we need to ensure that our groups are not pseudo-solvable. From the generic point of view adopted in this work, this is a corollary of the next proposition.

Proposition 5.3. *Consider* $\mathrm{Diff}^{\omega}(M)$ *equipped with its analytic topology. There is a residual set $\mathcal{R} \subset$* $\mathrm{Diff}^{\omega}(M) \times \mathrm{Diff}^{\omega}(M)$ *such that the group generated by a pair of diffeomorphisms (f_1, f_2) in $\mathcal{R}$ is free.*

Consider an irreducible word $W(a_1, a_2)$ in the symbols a_1, a_2, that is a formal list of the form

$$W(a_1, a_2) = a_{j_n}^{i_n}.a_{j_{n-1}}^{i_{n-1}}.\cdots.a_2^{i_2}.a_1^{i_1} \tag{1}$$

where $j_n = 1$ or 2 depending on n being odd or even and $i_k \in \mathbb{Z}^*$ (note that without loss of generality we can suppose that $W(a_1, a_2)$ starts with a_1). Denote by $W(f_1, f_2)$ the element of $\mathrm{Diff}^{\omega}(M)$ obtained by replacing a_1 by f_1 (resp. a_2 by f_2) and considering the "dot" between two consecutive $a_{l+1}^{i_{l+1}} a_l^{i_l}$ as composition. Because there are only countable many irreducible words $W(a_1, a_2)$, the standard Baire's argument allows us to conclude Proposition (5.3) from Proposition (5.4).

Proposition 5.4. *Fix $W(a_1, a_2)$ as above. The set $\mathcal{R}_W$ of pairs $(f_1, f_2) \in \mathrm{Diff}^{\omega}(M) \times \mathrm{Diff}^{\omega}(M)$ such that $W(f_1, f_2)$ is different from the identity is open and dense.*

Obviously the set $\mathcal{R}_W$ is open. Thus it is enough to check that it is also dense. We then consider an irreducible word $W(a_1, a_2)$ as in (1) and a pair (f_1, f_2) of diffeomorphims in $\mathrm{Diff}^{\omega}(M) \times \mathrm{Diff}^{\omega}(M)$ such that $W(f_1, f_2) = id$. Without loss of generality, we can suppose that W has *minimal length* among the irreducible words $W(a_1, a_2)$ as in (1) satisfying $W(f_1, f_2) = id$ (the length of an irreducible word as in (1) is by definition the value $i_1 + i_2 + \cdots + i_n$). We need to prove that f_1, f_2 can be perturbed in the analytic topology to provide elements $\hat{f}_1, \hat{f}_2 \in \mathrm{Diff}^{\omega}(M)$ such that $W(\hat{f}_1, \hat{f}_2) \neq id$.

Given a point $p \in M$, the *orbit under* $W(f_1, f_2)$ of p is by definition the finite set consisting of points of the form

$$\mathcal{O}_W(p) = \bigcup_{k=1}^{n} \bigcup_{1 \leq l_k \leq i_k} \left\{ f_{j_k}^{l_k} \circ \cdots \circ f_1^{i_1}(p) \right\}$$

where j_k equals 1 or 2 depending on k being odd or even. In particular for $k = 1$ the set $f_{j_k}^{l_k} \circ \cdots \circ f_1^{i_1}(p)$ is reduced to the points $f_1(p), f_1^2(p), \cdots, f_1^{i_1}(p)$. Because of the assumption that W has minimal length among irreducible words satisfying $W(f_1, f_2) = id$, we can select $p \in M$ such that $\mathcal{O}_W(p)$ consists of pairwise distinct points which will be numbered as $p_1, \ldots, p_{s-1}, p_s = p$. Consider also a neighborhood U of p such that the orbit of U under W, $\mathcal{O}_W(U)$, consists of open sets $\{U_1, \ldots, U_{s-1}, U_s = U\}$ pairwise disjoint and such that U_i is a neighborhood of p_i.

Consider a C^{∞} vector field X which vanishes identically on $M \setminus U_{s-1}$ and satisfy $X(p_{s-1}) \neq 0$. We denote by Φ_X the (global) flow generated by X. Replacing f_{j_n} by $f_{j_n} \circ \Phi_X^t$, we see that the orbit of p under W is affected only around $f_{j_n}^{i_n} \circ \cdots \circ f_1^{i_1}(p)$ (which previously agreed with p itself). Precisely the new orbit becomes $\{p_1, \ldots, p_{s-1}, p_s = f_{j_n} \circ \Phi_X^t \circ f_{j_n}^{i_n - 1} \circ \cdots \circ f_1^{i_1}(p)\}$. If t is small enough, we can ensure that $\Phi_X^t(p) \neq p$ so that $p_s \neq p$. Therefore we have constructed a C^{∞} perturbation of f_1, f_2 satisfying our requirements, namely such that $W(f_1, f_2) \neq id$. Now we have to turn this C^∞ perturbation into an analytic one. Here we shall use the argument of [B-T]. We keep the preceding notations.

Due to technical reasons, we embed X into the family $X_\lambda = \lambda.X$, $\lambda \in [0,1]$, of C^∞ vector fields vanishing identically on $M \setminus U_{s-1}$ for all λ. In particular X_0 vanishes identically on M and the derivative of $X_\lambda(p)$ with respect to λ at $\lambda = 0$, $\partial X_\lambda(p)/\partial\lambda \mid_{\lambda=0}$, is different from *zero*. Accordingly, we denote by $\Phi_{X,\lambda}$ the flow of X_λ. Finally let $t_0 > 0$ be so small that $\Phi^{t_0}_{X,\lambda}(p) \neq p$ for all $\lambda \in (0,1]$ (this assumption can always be made without loss of generality).

Assume we are given a neighborhood $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) of f_1 (resp. f_2) in the analytic topology. We need to find $\hat{f}_1$ in $\mathcal{U}_1$ (resp. $\hat{f}_2$ in $\mathcal{U}_2$) such that $W(\hat{f}_1, \hat{f}_2) \neq id$.

Let $X_{\lambda,T}$ be the vector field obtained by evolving X_λ under the heat equation on M (which is defined in accordance with the analytic metric induced on M through the original embedding of M into $\mathbb{R}^N$). It is proved in [B-T] that $X_{\lambda,T}$ is real analytic for all $T > 0$ and, besides, one has the estimates

$$\begin{aligned} &\| X_{\lambda,T} - X_\lambda \|_{C^k} = O(T^{1/4}) \, \| X_\lambda \|_{C^{k+1}} \quad \text{and} \\ &\| X_{\lambda,T} \|_\tau \leq \exp\left(\tau^2/4T\right) . \sup_{p\in M} \| X_\lambda(p) \| , \end{aligned} \tag{2}$$

where $\| \, . \, \|_{C^k}$ (resp. $\| \, . \, \|_{C^{k+1}}$) stands for the usual C^k (resp. C^{k+1}) norm and the constant is determined by the heat kernel.

Proof of Proposition (5.4). To simplify the notations, let us suppose that $j_n = 2$ (the case $j_n = 1$ is completely analogous). We denote by $f_{2,\lambda,T}$ the diffeomorphism $f_2 \circ \Phi^{t_0}_{\lambda,T}$. We want to check that it is possible to choose λ, T such that $f_{2,\lambda,T}$ belongs to $\mathcal{U}_2$ and $W(f_1, f_{2,\lambda,T}) \neq id$.

So consider the function $F(\lambda) = \| W(f_1, f_2 \circ \Phi^{t_0}_{X,\lambda})(p) - p \|$ and note that $\partial F/\partial\lambda \neq 0$ at $\lambda = 0$. Next we define the family of functions $F_T(\lambda) = \| W(f_1, f_{2,\lambda,T})(p) - p \|$. If F_T is a function sufficiently C^1-close to F (T fixed), then $\partial F_T/\partial\lambda$ is different from *zero* at $\lambda = 0$. This implies in turn the existence of arbitrarily small $\lambda > 0$ such that $F_T(\lambda) \neq 0$. Therefore, to conclude the proof, we take $T > 0$ so small that $\partial F_T(0)/\partial\lambda \neq 0$ which is possible in view of the first estimate in (2). Now, for such a fixed T, we can find $\lambda > 0$ small enough to ensure that $F_T(\lambda) \neq 0$ and $\| X_{\lambda,T} \|_\sigma < \delta$ for any previously fixed $\sigma, \delta > 0$. Finally if σ, δ are suitably chosen, it is clear that $f_{2,\lambda,T}$ lies in $\mathcal{U}_2$ as desired. This proves the proposition in question as well as Proposition (5.3). □

Now we shall prove Theorem D.

Proof of Theorem D. Denote by $\widetilde{M}$ a complexification of M and let G be a group as in the statement of the theorem. Using Proposition (5.3) we can suppose that G is a free group. Now, Ghys's theorem (Theorem (5.2)) ensures the existence of $\tau > 0$ and of a sequence of diffeomorphims $\{h_i\} \subset G$ ($h_i \neq id$ for all $i \in \mathbb{N}$) whose corresponding holomorphic extensions $\{\tilde{h}_i\}$ are defined on $\widetilde{M}_\tau$ and, in fact, converge uniformly to the identity on $\widetilde{M}_\tau$.

Let f be a Morse-Smale diffeomorphism belonging to G which exists since (f_1, f_2) belongs to $\mathcal{V}$. Modulo perturbing f with the techniques explained above, we can assume without loss of generality that all the attractors and all the repellers of f or of an iterate f^k are non-resonant. Recall that the non-wandering set $\Omega(f)$ of f is finite and therefore consists of a finite number of fixed and periodic points of f. Consider the points $p_1, \dots, p_r \in \Omega(f)$ of periods $s_1, \dots, s_r$ such that p_i is either a hyperbolic attractor (sink) or a hyperbolic repeller (source) for f^{s_i}. Fixed $i \in \{1, \dots, r\}$, denote by $\text{Bas}(f^{s_i}, p_i)$ (resp. $\text{Bas}(f^{-s_i}, p_i)$) the *basin* of p_i with respect to f^{s_i} (resp. f^{-s_i}), namely it is the set of points $x \in M$ such that the sequence $\{f^{s_i k}(x)\}_{k\in\mathbb{N}}$ (resp. $\{f^{-s_i k}(x)\}_{k\in\mathbb{N}}$) converges to p_i. It is easy to see that the union $\bigcup_{i=1}^r (\text{Bas}(f^{s_i}, p_i) \cup \text{Bas}(f^{-s_i}, p_i))$ covers M apart from a finite number of points belonging to $\Omega(f)$. However by considering an element $g \in G$ which is not a power of f (so that g, f generates a free subgroup on two generators), we can suppose that the g-orbit of the finitely many points in $M \setminus \bigcup_{i=1}^r (\text{Bas}(f^{s_i}, p_i) \cup \text{Bas}(f^{-s_i}, p_i))$ are infinite. In other words, the g-orbits of these points intersect $\bigcup_{i=1}^r (\text{Bas}(f^{s_i}, p_i) \cup \text{Bas}(f^{-s_i}, p_i))$.

Claim. Every point p in M possesses a neighborhood U_p equipped with a non-trivial vector field X which is in the C^∞-closure of G (cf. Definition (4.3)).

Proof of the Claim. After the above remark concerning the g-orbit of points belonging to $M \setminus \bigcup_{i=1}^r (\text{Bas}(f^{s_i}, p_i) \cup \text{Bas}(f^{-s_i}, p_i))$, it suffices to show that any point p in, say, $\text{Bas}(f^{s_1}, p_1)$ (p_1 being an attractor of f^{s_1}) possesses a neighborhood U equipped with a non-trivial vector field X having the required properties. Actually we just need to construct a non-trivial vector field X contained in the C^∞-closure of G and defined on a neighborhood of p_1. However, if $\widetilde{U} \subset \widetilde{M}_\tau$ is a neighborhood of p_1 in $\widetilde{M}$, then Proposition (2.1) ensures the existence of a vector field $\widetilde{X}$ defined on $\widetilde{U}$ (modulo reducing $\widetilde{U}$) with the property pointed out in the statement of this proposition. Indeed, p_1 is a hyperbolic attractor of f^{s_i} and f^{s_i} is C^ω-conjugate to its linear part on a neighborhood of p_1 (Poincaré's theorem, recall that f was already made generic in the sense that the attractors or repellers of the iterate f^{s_i} are not resonant). On the other hand, we have also the existence of a sequence of elements $\{\tilde{h}_i\}$ converging uniformly to the identity on $\widetilde{U}$ so that all the assumptions of Proposition (2.1) are verified. Finally Cauchy Formula shows that the restriction of $\widetilde{X}$ to $\widetilde{U} \cap M$ is in the C^{∞}-closure of G. To complete the proof of the claim we need to show that this restriction cannot vanish identically. For this we notice that $\widetilde{X}$ was constructed as a limit of diffeomorphisms that preserve the real manifold M. Thus, in local coordinates, the Taylor series of these diffeomorphisms have real coefficients. Then the same applies to the Taylor series of $\widetilde{X}$. It follows that, if $\widetilde{X}$ vanished identically on M (the real manifold), it would vanish identically on $\widetilde{U}$ as well. We have already seen that this is impossible so that the claim results at once. $\square$

The rest of the proof of Theorem is now standard. Under generic assumptions the group G has, in fact, many linearly independent vector fields X as above on a neighborhood of each point. The rest of the statement follows as in [L-R] (see also for [Be] for a slightly different argument). Further details are provided in the appendix in order to make the paper more self-contained.

6 Vector fields and proof of Theorem E

In this section we shall prove Theorem E. In the course of the proof we shall review the general construction of vector fields in the closure of groups (or pseudogroups) used in particular in the proof of Proposition (2.1). As mentioned our treatment is strongly inspired in our joint work with F. Loray [L-R]. However, in the present case, it is less technical and therefore becomes shorter and clearer compared to the discussion of [L-R]. As to the treatment given in [Be], our discussion seems to be applicable to a larger class of situations as suggested by Theorem E.

Denote by $\mathbb{B}^n \subset \mathbb{C}^n$ the unit ball of $\mathbb{C}^n$. Consider a pseudogroup Γ consisting of *holomorphic maps* from open subset of $\mathbb{C}^n$ to $\mathbb{C}^n$. In the course of this section we shall be involved with different uniform constants whose specific value is not important. They will be denoted by Const or const so that, sometimes, different constants will be assigned with the same name. The proposition below makes it clear what is the fundamental ingredient to construct vector fields as desired.

Proposition 6.1. *Suppose that Γ contains a sequence of maps H_i satisfying the assumptions below:*

$\imath$) *The maps H_i converge uniformly to the identity on a common open domain $U \subset \mathbb{B}^n$. Besides $H_i \neq Id$ for every $i \in \mathbb{N}$.*

$\imath\,\imath$) There is a relatively compact open subset $V \subset U$ and a uniform constant C_1 such that

$$\sup_U \|H_i(z) - z\| \;\leq\; C_1 \sup_V \|H_i(z) - z\| \,. \tag{3}$$

Then there is a non-trivial C^ω-vector field X in the closure of Γ relative to V (in the sense of Proposition (2.1)).

Proof. The proof is very simple. Let $\{H_i\}$ be a sequence of elements in Γ as in the statement. For each sufficiently large $i \in \mathbb{N}$, we consider the vector field

$$X_i(z) = \frac{1}{\sup_U \,||\, H_i(z) - z \,||} \mathrm{Vect}(H_i(z) - z)$$

which is defined on U (where $\mathrm{Vect}(H_i(z) - z)$ stands for the vector of extremities $H_i(z)$ and z). By construction we have $\sup_U \,||\, X_i(z) \,|| = 1$. Hence Montel Theorem ensures the existence of a subsequence (still denoted by $\{X_i\}$) converging uniformly on V towards a vector field X. Estimate (3) allows one to conclude that X is a non-trivial vector field since $\sup_V \,||\, X_i \,|| \geq 1/C_1 > 0$ for all $i \in \mathbb{N}$. Finally the standard argument of "polygonal approximations" used in Peano's theorem of existence of solutions for continuous differential equations (cf. for instance [Reb1]) shows that X possesses the desired properties. Namely the flow of X at time t is approximated by a sequence of the form $H_i^{n_i}(z)$ where n_i is given as the limit $[t/\sup_V \,||\, H_i(z) - z \,||]$ (up to passing to a subsequence and where $[\,.\,]$ stands for the integral part). The proposition is proved. □

Building on Proposition (6.1) we can give a short proof of Proposition (2.1) as well as a proof of Theorem E. Naturally both proofs boil down in finding sequences satisfying the conditions of Proposition (6.1) and contained in Γ (for Theorem E the sequence must be contained in a pseudogroup obtained by suitable restrictions of skew-products in G). Let us begin by proving Proposition (2.1).

Let us fix $k \in \mathbb{N}$ and consider the space $P(k, \mathbb{C}^n)$ of maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ whose components are polynomials of degree at most k on the coordinates $z_1, \ldots, z_n$ of $\mathbb{C}^n$. Consider also open sets $U_1 \subseteq U_2$ of $\mathbb{C}^n$ and a constant $C_1 > 0$. The following lemma is elementary and relies on the fact that maps in $P(k, \mathbb{C}^n)$ have only finitely many "coefficients" (more "abstractly" it relies on the local compactness of this space).

Lemma 6.2. *Let U_1, U_2 be as above. Then there is a constant C_1 such that*

$$\sup_{U_2} \,||\, f(z) - z \,|| \leq C_1 \sup_{U_1} \,||\, f(z) - z \,|| \;\; .$$

for every $f \in P(k, \mathbb{C}^n)$.

Proof. Suppose for a contradiction that the statement is false. Let $\{f_i\} \subset P(k, \mathbb{C}^n)$ be a sequence such that

$$\lim_{i\to\infty} \frac{\sup_{U_2} \| f_i(z) - z \|}{\sup_{U_1} \| f_i(z) - z \|} = \infty \,. \tag{4}$$

Note that we can suppose without loss of generality that the maximum of the modulus of the coefficients of $f_i(z) - z$ is 1. Hence it follows that the sequence $\{f_i\}$ is uniformly bounded on U_2. Thus it admits a subsequence which is uniformly convergent in U_1. Let f_∞ be one such uniform limit. Since $\{f_i\}$ is uniformly bounded, Equation (4) implies that f_∞ must coincide with the identity. However since all the maps are in $P(k, \mathbb{C}^n)$ (i.e. have only finitely many coefficients), Cauchy Formula guarantees that uniform convergence is equivalent to convergence of all coefficients. It follows that all the coefficients of $f_\infty(z) - z$ are *zero* and that they are limits of the corresponding coefficients of $f_i(z) - z$. This is however impossible since all the maps $f_i(z) - z$ have at least one coefficient of modulus 1. The resulting contradiction establishes the lemma. □

Now we fix Γ as in the statement of Proposition (2.1). Our sets U, V will be chosen as balls centered at the origin and with appropriate radii. We begin the construction of the desired sequence H_i by noticing the existence of $\delta > 0$ with the following property:

$(*)$ if h is a differentiable map defined on $B(4\lambda_1/5)$ and such that $\sup_{\mathbb{B}(4\lambda_1/5)} \| h(z) - z \| < \delta$, then the map $F^{-1} \circ h \circ F$ is defined on $B(4\lambda_1/5)$ as well.

For each sufficiently large $i \in \mathbb{N}$, we consider the maps $F^{-1} \circ h_i \circ F$, $\cdots$, $F^{-j(i)} \circ h_i \circ F^{j(i)}$ where $j(i)$ is the greatest positive integer for which the map $F^{-j(i)} \circ h_i \circ F^{j(i)}$ is defined on $B(4\lambda_1/5)$ ($j(i) = \infty$ if all the maps $F^{-j} \circ h_i \circ F^j$, $j \in \mathbb{N}$, are defined on $B(4\lambda_1/5)$).

The construction of the sequence H_i will be divided into two cases.

Case 1: If, modulo passing to a subsequence, one has $j(i) < \infty$ for every i.

By virtue of the definition of $j(i)$ and of observation $(*)$, it follows that

$$\sup_{\mathbb{B}(3\lambda_1/4)} \| F^{-j(i)} \circ h_i \circ F^{j(i)}(z) - z \| \geq \delta > 0 \,.$$

Next we fix a sequence $\{\delta_r\}_{r\in\mathbb{N}}$ of reals $\delta_1 > \delta_2 > \cdots > 0$ converging to *zero* when r goes ∞. Fixed i, r, we define $j(i, r)$ as the smallest positive integer for which

$$\sup_{\mathbb{B}(3\lambda_1/4)} \| F^{-j} \circ h_i \circ F^j(z) - z \| \geq \delta_r > 0 \,. \tag{5}$$

Note that the existence of $j(i,r)$ is ensured by the fact that $j(i) < \infty$. Finally let $k \in \mathbb{N}$ be the smallest positive integer such that $|\lambda_n|^k < |\lambda_1|$.

Construction of the sequence H_i in Case 1. The Taylor expansion of $(F^{-j(i,r)} \circ h_i \circ F^{j(i,r)} - id)$ gives

$$\begin{aligned} F^{-j(i,r)} &\circ h_i \circ F^{j(i,r)}(z) - z = \\ &= F^{-j(i,r)} \circ h_i \circ F^{j(i,r)}(0) + (D_0(F^{-j(i,r)} \circ h_i \circ F^{j(i,r)}) - I)z + \\ &\quad + \cdots + D_0^k(F^{-j(i,r)} \circ h_i \circ F^{j(i,r)})z^k + R(j(i,r), z) = \\ &= \mathrm{Pol}_{j,i,r}(k,z) + R(j(i,r),z) \end{aligned}$$

where $R(j(i,r),z) \leq Const(\lambda_n^k/\lambda_1)^{j(i,r)}$ (where the constant $Const$ does not depend on j, i, r or z). Because $j(i,r)$ obviously goes to infinity as i goes to infinity for r fixed, it results that $R(j(i,r),z)$ converges uniformly towards *zero* on $\mathbb{B}(3\lambda_1/4)$ (recall that $|\lambda_n^k/\lambda_1| < 1$ for r fixed). Thus for large enough i, one has

$$\| (F^{-j(i,r)} \circ h_i \circ F^{j(i,r)}(z) - z) - \mathrm{Pol}_{j,i,r}(k,z) \| \leq \delta_r/10C_2 \,.$$

The statement now follows from Lemma (6.2). □

Case 2: There is $i_0 \in \mathbb{N}$ such that $j(i) = \infty$ for every $i > i_0$.

Note that the assumption above implies in particular that $h_i(0) = 0$ for large enough i.

Construction of the sequence H_i in Case 2. The Taylor series of the function $(h_i - id)$ based at the origin is given by

$$(h_i - id)(z) = h_i(z) + (D_0 h_i - I)z + \cdots$$

(where I stands for the identity matrix). Among the multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ ($\alpha_i \in \mathbb{N}$) such that $D_0^\alpha(h_i - id) \neq 0$, we consider those for which $\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}/\lambda_l$ attains its maximal value which will be denoted by Λ_i (where $l = 1, \ldots, n$ corresponds to the component of h_i in question). Clearly the value Λ_i can be attained only by finitely many multi-indices α.

To construct the sequence H_i in Case 2, we first need to consider the sequence $\{F^{-j} \circ h_i \circ F^j\}_{j \in \mathbb{N}}$, $i \in \mathbb{N}$ fixed, with regard to Equation (3). Precisely we want to prove that the above sequence satisfies this estimate. Because of the homogeneity of Equation (3), the last claim amounts to verify the existence of a constant $Const$ such that

$$\sup_{\mathbb{B}(3\lambda_1/4)} \| \Lambda_i^{-j}(F^{-j} \circ h_i \circ F^j(z) - z) \| \leq Const \sup_{\mathbb{B}(\lambda_1/2)} \| \Lambda_i^{-j}(F^{-j} \circ h_i \circ F^j(z) - z) \| .$$

Let $\overline{\alpha}_1^i, \ldots, \overline{\alpha}_s^i$ ($s \geq 1$) be the multi-indices for which $\lambda_1^{\alpha_1^i} \cdots \lambda_n^{\alpha_n^i}/\lambda_l$ equals its maximal value Λ_i ($\overline{\alpha}^i = (\alpha_1^i, \ldots, \alpha_n^i)$). Replacing $F^j(z) = (\lambda_1^j z_1, \ldots, \lambda_n^j z_n)$ in the Taylor expansion of h_i we obtain

$$\Lambda_i^{-j}(F^{-j} \circ h_i \circ F^j(z) - z) = c_{\overline{\alpha}_1^i} z^{|\overline{\alpha}_1^i|} + \cdots + c_{\overline{\alpha}_s^i} z^{|\overline{\alpha}_s^i|} + R_i(j, z) \qquad (6)$$

where $|\overline{\alpha}| = \alpha_1 + \cdots + \alpha_n$ and $R_i(j, z)$ goes uniformly to *zero* when $j \to \infty$ (i fixed). By definition $c_{\overline{\alpha}_1^i}, \ldots, c_{\overline{\alpha}_s^i} \neq 0$, so that Lemma (6.2) implies that the sequence $\{F^{-j} \circ h_i \circ F^j\}_{j \in \mathbb{N}}$, i fixed, verifies an estimate like (3) given that $R_i(j, z)$ converges uniformly to *zero*.

Note that the construction of the sequence will be completed if $\{F^{-j} \circ h_{i_0} \circ F^j\}_{j \in \mathbb{N}}$ converges to the identity for some $i_0 \in \mathbb{N}$. In particular if $\Lambda_{i_0} < 1$ for some i_0 then Estimate (6) ensures this convergence and establishes the lemma. On the other hand, we always have $\Lambda_i \leq 1$, otherwise $j(i)$ would be finite.

So we assume that $\Lambda_i = 1$ for every $i \in \mathbb{N}$. In this case the norm $| \ |$ of the multi-indices $\overline{\alpha}^i$'s for which $\lambda_1^{\alpha_1^i} \cdots \lambda_n^{\alpha_n^i}/\lambda_l = 1$ is uniformly bounded (i.e. the value of $|\overline{\alpha}_r^i|$ appearing in equation (6) is bounded by a constant which does not depend on $i \in \mathbb{N}$). Since the coefficients $c_{\overline{\alpha}_s^i}$ are coefficients of the Taylor series of $h_i - id$, Cauchy's formula implies that these coefficients converge to *zero* when i increases (since h_i converges to the identity and the degree associated to these coefficients is uniformly bounded). So we just need to associate to each $i \in \mathbb{N}$ a sufficiently large integer $j(i) \in \mathbb{N}$ in order to guarantee that the resulting sequence $\{F^{-j(i)} \circ h_i \circ F^{j(i)}\}_{i \in \mathbb{N}}$ fulfils all the required conditions. The proof of Proposition (2.1) is finally over. □

We now proceed to the proof of Theorem E. Recall that $\mathcal{G}$ (resp. $\mathcal{G}_{\mathbb{C}}$) is the group of skew-products over $\mathrm{PSL}(2, \mathbb{R})$ (resp. $\mathrm{PSL}(2, \mathbb{C})$) mentioned in Section 2. It is enough to deal with $\mathcal{G}$ since all proofs apply word-by-word to the case of $\mathcal{G}_{\mathbb{C}}$. We want to prove the theorem below.

Theorem 6.3. *Let $G \subset \mathcal{G}$ be as in the statement of Theorem E. Then there exists an open interval $I \subset \mathbb{S}^1$ and a non-trivial C^ω-vector field X on $V = I \times \mathbb{R}$ which is the closure of Γ_G relative to V, where Γ_G stands for the pseudogroup induced by the restrictions to V of the elements in G.*

First we observe that $\mathsf{G}_\pi \subset \mathrm{PSL}(2, \mathbb{R})$ (the projection of G on the first coordinate) is not solvable. Indeed, if it were solvable it would, in fact, be Abelian. Hence the first derived subgroup of G would consist of elements having the form $(x, y) \mapsto (x, y + w(x))$. Such a group is clearly Abelian thus implying that G is step-2 solvable. Therefore G_π is neither Solvable nor discrete (the latter conclusion being part of our assumptions). It follows from elementary considerations regarding the Lie group $\mathrm{PSL}(2, \mathbb{R})$ and its Lie algebra that G_π is, in fact, dense in $\mathrm{PSL}(2, \mathbb{R})$. In particular we can find a hyperbolic element $A \subset \mathsf{G}_\pi$. Since A is hyperbolic, we can suppose without loss of generality that the north-pole p_N is an attracting fixed point of A. Next we consider a (not necessarily unique) element $F \subset \mathsf{G}$ which has the form $F(x, y) = (Ax, y + u(x))$ for some C^ω-function $u : \mathbb{S}^1 \to \mathbb{R}$ (note that our additive notation leads us to work on $\mathbb{S}^1 \times \mathbb{R}$ with the second coordinate viewed as the universal covering of $\mathbb{S}^1$). Notice also that, in the statement below, there is no reference to whether or not the functions appearing in the second coordinate of the generators of G are close to *zero*.

Consider a complexification $\widetilde{\mathbb{S}^1}$ of $\mathbb{S}^1$. One has:

Lemma 6.4. *There is a neighborhood $\mathcal{U} \subset \widetilde{\mathbb{S}^1}$ of $\mathbb{S}^1$ and a sequence of skew-products $\{h_k\} \subset \mathsf{G}$ satisfying the following:*

1. Each h_k has the form $h_k(x, y) = (B_k x, y + v_k(x))$ where B_k, v_k admit a holomorphic extension to $\mathcal{U}$.

2. Denoting by $\widetilde{B}_k$ (resp. $\widetilde{v}_k$) the holomorphic extensions of the B_k's (resp. v_k's) to $\mathcal{U}$, it follows that $\{\widetilde{B}_k\}$ (resp. $\{\widetilde{v}_k\}$) converges uniformly towards the identity (resp. zero) on $\mathcal{U}$.

Proof. Although our notation seems to involve a non-compact space, namely $\mathbb{S}^1 \times \mathbb{R}$, the statement is actually a (much easier) particular case of Theorem (5.2). In fact, it suffices to notice that the skew-products in $\mathcal{G}$ commute with the "vertical translations" $(x, y) \mapsto (x, y + \text{const.})$. Clearly this allows us to restrict the problem to a compact part of $\mathbb{S}^1 \times \mathbb{R}$. Also it is clear that, in the present case, the convergence to the identity on the first coordinates of a sequence of commutators implies the same convergence for the corresponding second coordinates. $\square$

Let $I \subset \mathbb{S}^1$ be an interval containing the north-pole p_N. Suppose that we have a sequence $\{H_i\} \subset \mathsf{G}$, $H_i(x, y) = (C_i x, y + w_i(x))$, such that all the C_i's and all the w_i's admit a holomorphic extension to an open set $\tilde{I} \subset \widetilde{\mathbb{S}^1}$ of I ($\tilde{I} \cap \mathbb{S}^1 = I$). Suppose also that these holomorphic extensions converge uniformly respectively to the identity and to zero on $\tilde{I}$. Let $I_0 \subset I$ be a relatively compact sub-interval of I and let $\tilde{I}_0$ be an open neighborhood if I_0 which is compactly contained in

$\tilde{I}$. Thanks to Proposition (6.1), the proof of Theorem (6.3) is reduced to find a sequence $\{H_i\}$ as above and a uniform constant Const such that

$$\text{Const.} \sup_{\tilde{I}_0} \|\widetilde{w}_i(x)\| \geq \sup_{\tilde{I}} \|\widetilde{w}_i(x)\| , \tag{7}$$

where $\widetilde{w}_i$ stands for the holomorphic extension of w_i to $\tilde{I}$. Indeed the sequence $\{C_i\}$ formed by the first coordinates of the H_i's necessarily satisfies an estimate similar to estimate (7) as follows from the fact that all the C_i's belong to a compact neighborhood of the identity in $\mathrm{PSL}(2,\mathbb{R})$ (this is essentially the same local compactness argument employed in Lemma (6.2)). Therefore the sequence $\{H_i\} \subset \mathsf{G}$ verifies the assumptions of Proposition (6.1) so that Theorem (6.3) results at once.

In practice the desired sequence $\{H_i\}$ will be obtained from the sequence $\{h_k\}$ of Lemma (6.4) by means of a procedure similar to the procedure used in the proof of Proposition (2.1). Recall that $h_k(x, y) = (B_k x, y + v_k(x))$. The group G also contains a skew-product $F(x, y) = (Ax, y + u(x))$ such that the north-pole p_N is an attracting fixed point of A. Given $n \in \mathbb{N}$, we have

$$F^n(x, y) = (A^n x\ ,\ y + \sum_{j=0}^{n-1} u(A^j x) \quad \text{and} \quad F^{-n}(x, y) = (A^{-n}x, y - \sum_{j=1}^{n} u(A^{-j}x))\,.$$

Hence, fixed $n,\ k$,

$$F^{-n} \circ h_k \circ F^n = (A^{-n} B_k A^n x, y + v_k(A^n x) + \sum_{j=0}^{n-1} u(A^j x) - \sum_{j=1}^{n} u(A^{-j} B_k A^n x))\,.$$

Defining $w^{kn}(x) = v_k(A^n x) + \sum_{j=0}^{n-1} u(A^j x) - \sum_{j=1}^{n} u(A^{-j} B_k A^n x)$ it is enough to find a sequence of pairs (k, w) such that the corresponding sequence $\{w^{kn}\}$ satisfies Estimate (7).

In appropriate local coordinates around p_N ($p_N \simeq 0$), we can suppose that $Ax = \lambda x$ for some $\lambda \in (0, 1)$. We also set $B_k(x) = b_k^0 + b_k^1 x +$ h.o.t. with $b_k^s \in \mathbb{R}$. In these coordinates w^{kn} becomes

$$\begin{aligned} w^{kn}(x) &= v_k(\lambda^n x) + \sum_{j=0}^{n-1} u(\lambda^j x) - \sum_{j=1}^{n} u(\lambda^{-j} B_k(\lambda^n x)) \\ &= v_k(\lambda^n x) - \sum_{j=1}^{n} u \circ \lambda^{n-j} \circ (\lambda^{-n} B_k \lambda^n - Id)(x)\,. \end{aligned} \tag{8}$$

The next step is to introduce a relation between k, n. Modulo passing to a subsequence of the elements $h_k = (B_k x, y + v_k(x))$, we can suppose without loss of generality that $B_k(0) \neq 0$ (where $0 \simeq p_N$). Indeed suppose that $B_k(0) = 0$ for all but finitely many integers k. Since G_π is dense in $\mathrm{PSL}(2, \mathbb{R})$, the conjugates $D^{-1} B_k D$ for a generic element $D \in \mathrm{PSL}(2, \mathbb{R})$ (i.e. D such that $D(0) \neq 0$) will not fix 0 any longer and will still provide a sequence of elements of G satisfying the preceding conditions (modulo shirinking domains). Now we introduce an auxiliary metric in $\mathrm{PSL}(2, \mathbb{R})$ and fix a small neighborhood $U(\epsilon)$ of the identity in $\mathrm{PSL}(2, \mathbb{R})$. Clearly B_k belongs to $U(\epsilon)$ for sufficiently large k. Given $B_k \in U(\epsilon)$, we define $n(k)$ as the smallest positive integer for which $A^{-n(k)} B_k A^{n(k)}$ does not belong to $U(\epsilon)$.

Lemma 6.5. *For every k large enough there is $n(k)$ such that $A^{-n(k)} B_k A^{n(k)}$ does not belong to $U(\epsilon)$. Furthermore $A^{-n(k)} B_k A^{n(k)}$ still belongs to a neighborhood $U(r\epsilon)$ of the identity in* $\mathrm{PSL}(2, \mathbb{R})$ *for some constant r.*

Proof. Consider the local coordinate x mentioned above in which $p_N \simeq 0$ and $Ax = \lambda x$. Let $B_k = b_k^0 + b_k^1 x + \cdots$ with $b_k^0 \neq 0$. Hence

$$A^{-n} B_k A^n(x) = \lambda^{-n} b_k^0 + b_k^1 x + \lambda^n (b_k^2 x + \cdots)\,.$$

In particular $A^{-n} B_k A^n(0) = \lambda^{-n} b_k^0$ so that $A^{-n} B_k A^n$ lies in the complement of $U(\epsilon)$ for n very large. For the second part of the statement note that $A^{-n(k)+1}$ $B_k A^{n(k)-1}$ belongs to $U(\epsilon)$ by definition of $n(k)$. Thus $A^{-n} B_k A^n$ must belong to the neighborhood $U(r\epsilon)$ (for some r which does not depend on ϵ) given by $A^{-1} U(\epsilon) A$. The proof of the lemma is over. □

The last ingredient can be presented as follows. Given k and r, $0 \leq r \leq n(k)$, we consider the vector $\mathrm{vec}_{kr}(x)$ defined by setting the point x as its origin and the point $\lambda^{-r} B_k \lambda^r x$ as its final extremity. In classical vectorial notation we have $\mathrm{vec}_{kr}(x) = \lambda^{-r} B_k \lambda^r x - x$. On the other hand, let us recall that there is a uniform constant const > 0 such that

$$\sup_{\tilde{I}} \|B_k(x) - x\| \geq \sup_{\tilde{I}_0} \|B_k(x) - x\| \geq \text{const} \sup_{\tilde{I}} \|B_k(x) - x\|\,.$$

In fact, the above estimate results once again from the fact that all the B_k's belong to a compact part of $\mathrm{PSL}(2, \mathbb{R})$. This estimate immediately leads to

$$\begin{aligned} \lambda^{n(k)-r} \sup_{\tilde{I}} \|\mathrm{vec}_{kn(k)}(x)\| &\geq \sup_{\tilde{I}_0} \|\mathrm{vec}_{kr}(x)\| \\ &\geq \text{const}\, \lambda^{n(k)-r} \lambda^{n(k)-r} \sup_{\tilde{I}} \|\mathrm{vec}_{kn(k)}(x)\| \end{aligned} \tag{9}$$

for every k and $0 \leq r \leq n(k)$.

Proof of Theorem (6.3). As mentioned we just need to find a sequence $\{H_i\} \subset \mathsf{G}$ satisfying conditions of Proposition (6.1). Fixing $\varepsilon > 0$, the preceding amounts to find $H = (Cx, y + w(x)) \in \mathsf{G}$ such that $w(x)$ satisfies (7) for a uniform constant as indicated and

$$\max \ \{\sup_{\tilde{I}} \|Cx - x\| \ , \ \sup_{\tilde{I}} \|w(x)\|\} < \varepsilon \, .$$

We are going to prove that, modulo choosing the preceding k very large and the preceding ϵ very small, there is an element $H \in \mathsf{G}$ verifying the conditions in question. The statement will then follow as a consequence. We let $H = F^{-n(k)} \circ h_k \circ F^{n(k)} = (Cx, y = w(x))$ where k (and ϵ) will be chosen later on. According to Formula (8) we have

$$w(x) = v_k(x) - \sum_{j=1}^{n(k)} u \circ \lambda^{n(k)-j}(\lambda^{-j} B_k \lambda^j x - x) \ .$$

Now Taylor's expansion provides

$$\sum_{j=1}^{n(k)} u \circ \lambda^{n(k)-j}(\lambda^{-j} B_k \lambda^j x - x) = \sum_{j=1}^{n(k)} D_x(u \circ \lambda^{n(k)-j})\mathrm{vec}_{kj}(x) + R(x) \ .$$

However $D_x^2(u \circ \lambda^{n(k)-j}) = \lambda^{2(n(k)-j)} D^2_{\lambda^{n(k)-j}x} u$. Thanks to estimate (9), one has

$$\begin{aligned} \|R(x)\| &\leq \ \sup_{\tilde{I}} \| \ \mathrm{vec}_{kn(k)}(x)\|^2 \sum_{j=1}^{n(k)} \lambda^{3(n(k)-j)} D^2_{\lambda^{n(k)-j}x} u \\ &\leq \ \mathrm{Const} \ \sup_{\tilde{I}} \| \ \mathrm{vec}_{kn(k)}(x)\|^2 \sup_{\tilde{I}} \|D^2 u\| \, . \end{aligned} \tag{10}$$

On the other hand, using again the fact that G_π is dense in $\mathrm{PSL}(2, \mathbb{R})$, we can suppose without loss of generality that $D_0 u \neq 0$. Let $M = D_0 u \neq 0$. Modulo choosing ϵ very small and reducing I, $\tilde{I}$, we can suppose that $\sup_{\tilde{I}} \|D_x u\| \leq 3M/2$ and $\inf_{\tilde{I}} \|D_x u\| \geq M/2$. Recall that

$$w(x) - v_k(x) = \sum_{j=1}^{n(k)} u \circ \lambda^{n(k)-j}(\lambda^{-j} B_k \lambda^j x - x)$$

Hence, putting together all the estimates above, we conclude that

$$
\begin{aligned}
w(x) - v_k(x) \leq \frac{3M}{2} \sup_{\tilde{I}} \| \operatorname{vec}_{kn(k)}(x) \| \sum_{j=1}^{n(k)} \lambda^{n(k)-j} + \\
+ \text{ Const } \sup_{\tilde{I}} \| D^2 u \| \sup_{\tilde{I}} \| \operatorname{vec}_{kn(k)}(x) \|^2
\end{aligned}
\tag{11}
$$

as well as

$$
\begin{aligned}
w(x) - v_k(x) \geq \frac{M}{2} \text{const } \sup_{\tilde{I}} \| \operatorname{vec}_{kn(k)}(x) \| \sum_{j=1}^{n(k)} \lambda^{n(k)-j} \\
- \text{ Const } \sup_{\tilde{I}} \| D^2 u \| \sup_{\tilde{I}} \| \operatorname{vec}_{kn(k)}(x) \|^2 .
\end{aligned}
\tag{12}
$$

Up to reducing ϵ we can make $\|\operatorname{vec}_{kn(k)}(x)\|$ arbitrarily small. We then fix ϵ small enough to have

$$
\begin{aligned}
\| w(x) - v_k(x) \| &= \| \sum_{j=1}^{n(k)} u \circ \lambda^{n(k)-j} (\lambda^{-j} B_k \lambda^j x - x) \| \\
&\geq \frac{M}{4} \text{const} \left(\sum_{j=1}^{n(k)} \lambda^{n(k)-j} \right) \sup_{\tilde{I}} \| \operatorname{vec}_{kn(k)}(x) \| .
\end{aligned}
$$

Once ϵ as above is fixed, modulo choosing k very large, $\sup_{\tilde{I}} \|v_k(x)\|$ can be made arbitrarily small compared to $\sup_{\tilde{I}} \|\operatorname{vec}_{kn(k)}(x)\|$ since the latter is uniformly bounded from below in terms of I, $\tilde{I}$ and ϵ. The theorem then follows from Estimates (11) and (12). □

Proof of Theorem E. The argument is now similar to the proof of Theorem D. More details on the nature of these arguments are provided in the appendix. □

We want to close this article with some questions. It seems clear that the results presented here are not sharp (except maybe for Theorem A and, in some sense, for Theorem E). As to Theorem B and Corollary C, the most immediate and interesting question is to decide about the ergodicity of the generic action (recall that we have established only the existence of a dense orbit). Note also that the context of these results seems to exhibit some rigidity properties similar to the one discussed in Theorem D.

Other difficult question concerning these results is their possible extension to $C^2, C^3, \ldots$ topology. This seems much harder since KAM theory does not provide counter examples. On the other hand, any attempt to provide an affirmative answer apparently depends on a better understanding of questions related to the "Closing Lemma".

Still considering C^1-topology, it may be more reasonable to try to extend Corollary C to higher dimensions. In the volume-preserving case, there are some well-known difficulties in generalizing the theorem of Newhouse used in our proof. However even these difficulties might be more treatable in our context of non-Abelian free actions. In the symplectic case, M.-C. Arnaud has extended Newhouse's theorem to dimension 4 (cf. [Ar]). It might be interesting to check if our techniques combined with Arnaud's theorem would yield the existence of a dense orbit for the corresponding generic actions.

Concerning the second part of this paper, it is clear that the main result which need to be extended further is Theorem D. A minor point in improving this result is to dispense with the non-resonance condition on the eigenvalues of the Morse-Smale diffeomorphism. Indeed, this condition seems to be only a technical requirement to make our proofs simpler. Much more important is to dispense with the assumption regarding the existence of a Morse-Smale diffeomorphism. It is also interesting to extend these theorems to C^∞-topology (in which case we may keep the assumption on the existence of a Morse-Smale dynamics).

On the other hand, we may wonder whether or not "generic assumptions" would guarantee the existence of a diffeomorphism displaying a hyperbolic attractor fixed point. This may be a hard, but treatable, question. In fact, if the answer turns out to be affirmative, then the method of Section 6 should allow some new results about these generic dynamics. Another possibility of making some progress is to replace the existence of a Morse-Smale diffeomorphism by other classical type of dynamics, like partial hyperbolic ones.

Ultimately, one might want to consider the idea of a complete generalization of Theorem D. This question definitely seems to be very hard whereas it was successfully settled for $\mathrm{Diff}(\mathbb{C}, 0)$ (cf. [Na]) and $\mathrm{Diff}^\omega(S^1)$ (cf. [Reb1]).

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7 Appendix: Complement to the proof of Theorem D

In this appendix we shall provide additional details on the proof of Theorem D. It is easy to adapt the following arguments to derive Theorem E from Theorem (6.3). The discussion below also explain at least the part concerning topological rigidity of Corollary F. As mentioned the proof of "measurable rigidity" in Corollary F is a variant of the argument in [Reb2].

Consider the subset $\mathcal{V} \cap (\mathcal{U} \times \mathcal{U})$ of $\mathrm{Diff}^{\omega}(M) \times \mathrm{Diff}^{\omega}(M)$ described in the statement of this theorem. Let (f_1, f_2) be an element of $\mathcal{V} \cap (\mathcal{U} \times \mathcal{U})$. By virtue of Proposition (5.3) and, more generally, of the techniques of perturbation discussed in Section 5, without loss of generality we can suppose that:

1. The group $G \subset \mathrm{Diff}^{\omega}(M)$ generated by (f_1, f_2) is free;
2. G contains a non-resonant Morse-Smale diffeomorphism F;
3. Let p be a periodic point of F with period s. Writing F^s as a word $W_1(f_1, f_2)$, the orbit $\mathcal{O}_{W_1}(p)$ under $W_1(f_1, f_2)$ consist of pairwise distinct points (cf. notation of Section 5). Moreover the orbit of these points under G is infinite.
4. If p is an attractor of F^s (resp. F^{-s}), then we suppose that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of D_pF^s (resp. D_pF^{-s}) satisfy $0 <| \lambda_1 |< \cdots <| \lambda_n |< 1$. Moreover the maximum of the sets of the numbers of the form $\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}/\lambda_l$ is attained by a unique element (where $\alpha_i \in \mathbb{N}$ and $l \in \{1, \ldots, n\}$).
5. If h in G has only finitely many periodic points, then all these points are hyperbolic. Furthermore, if such a point is a hyperbolic attractor (resp. repeller) for a convenient iterate of h, we also assume that the corresponding eigenvalues are distinct and non-resonant.

In the sequel we always assume that (f_1, f_2) satisfies the above conditions. Note in particular that conditions 2, 3 and 4 are not only generic but also open on f_1, f_2. We denote by $\mathcal{K}_1 \subset \mathcal{V} \cap (\mathcal{U} \times \mathcal{U})$ the set of the pairs (f_1, f_2) satisfying these conditions. Clearly $\mathcal{K}_1$ is a residual set of $\mathcal{V} \cap (\mathcal{U} \times \mathcal{U})$. We also want to point out that it is possible to prove the Theorem D without using all these assumptions, however this would require a longer discussion and would not lead to a qualitative improvement in its statement.

Our first lemma is very easy and shows that it is enough to study a neighborhood of the attracting (resp. repelling) periodic points in $\Omega(F)$.

Lemma 7.1. *Assume we are given an arbitrary point $q \in M$. Assume also that, for each attracting (resp. reppeling) fixed point p_i of F^s (for some $s \in \mathbb{N}$), we are given a neighborhood U_i of p_i. Then the orbit of q under G intersects at least one of the U_i's.*

Proof. Just observe that the orbit under F of every point in the complement of $\Omega(F)$ satisfy the desired condition. On the other hand, because of Condition 3, if q belongs to $\Omega(F)$, then there is $h \in G$ such that $h(q)$ lies in $M \setminus \Omega(F)$. This concludes the proof. □

The following proposition will imply items 1 and 2 of Theorem D.

Proposition 7.2. *Let G be the subgroup generated by diffeomorphisms f_1, f_2 in $\mathrm{Diff}^\omega(M)$ and consider a Morse-Smale diffeomorphism $F \in G$ satisfying the above conditions. Suppose that $(f_1, f_2) \in \mathcal{K}_1$ are "sufficiently generic" (i.e. (f_1, f_2) will belong to a residual set $\mathcal{K}_2$ contained in $\mathcal{K}_1$). Then for each attracting (resp. repelling) fixed point p of F^s ($s \in \mathbb{N}^*$), there exists a neighborhood U of p endowed with n vector fields $X_1, \ldots, X_n$ which belong to the C^∞-closure of G and are linearly independent at any point of U (where n stands for the dimension of M).*

First consider a non-trivial analytic vector field Y^1 contained in the C^∞-closure of G and defined around $p \in M$ (without loss of generality we suppose that p is an attractor of F^s). We fix a local coordinate ϕ, identifying p with the origin of $\mathbb{R}^n \subset \mathbb{C}^n$, in which $F^s(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n)$. We want to prove that it is possible to find Y^1 as above such that $Y^1(0)$ has all components different from *zero* in the coordinate ϕ provided that f_1, f_2 are sufficiently generic.

Recall that the set of vector fields in the closure of G is invariant under multiplications by scalars. In other words, if Y belongs to the closure of G then aY belongs to this closure for any $a \in \mathbb{R}$. We claim that it is possible to suppose that Y^1 is given in the coordinate ϕ by $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial/\partial x_r$ where $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$ and $r \in \{1, \ldots, n\}$. To check the claim, just observe that the sequence of vector fields $(F^s)^* Y^1$ converges (after multiplication by an appropriate scalar) towards a vector field having the indicated form thanks to Condition 4 on the eigenvalues of $D_p F^s$. In particular the singular set of Y^1 is contained in the union of the Cartesian planes.

Part of the difficulty to prove Proposition (7.2) results from the fact that F changes when we perturb f_1, f_2. Thus we shall be led to use a scheme of perturbation similar to the one employed in the proof of Proposition (5.4). Our aim will be to establish the lemma below.

Lemma 7.3. *If f_1, f_2 are chosen sufficiently generic, then there is Y^2 in the closure of G, defined on a neighborhood of p and such that $Y^2(p)$ has all components different from zero when represented in the linearizing coordinate ϕ.*

In the sequel we resume the notations of Section 5. Recall that F^s is given as a word $W_1(f_1, f_2)$ in f_1, f_2. Moreover, because of Condition 3, the orbit of p under $W_1(f_1, f_2)$, $\mathcal{O}_{W_1}(p)$ consists of pairwise distinct points. Let V^p be a small neighborhood of p such that $\mathcal{O}_{W_1}(V^p)$ still consists of pairwise disjoint open sets.

Next, by iterating commutators as in Theorem (5.2), we can find $h = W_2(f_1, f_2)$ in G such that $h(p) \in V^p$ and, besides, there is $p^W \in \mathcal{O}_{W_2}(p)$ which lies on the complement of $\mathcal{O}_{W_1}(V)$ (here we use again Condition 3).

Proof of Lemma (7.3). We consider a neighborhood U^{p^W} of p^W such that $U^{p^W} \cap \mathcal{O}_{W_1}(V^p) = \emptyset$. Notice that, the restriction of F^s to V^p remains unchanged when f_1, f_2 are C^∞-perturbed *inside* U^p. Besides the restriction of Y^1 to V^p remains unchanged as well. Considering V^p in the coordinate ϕ, we have $Y^1 |_{V^p} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial/\partial x_r$ and $F^s |_{V^p} = (\lambda_1 x_1, \dots, \lambda_n x_n)$ Thus $h^* Y^1$ obviously provides a vector field defined around p and having all components at p different from *zero* in the coordinate ϕ as long as f_1, f_2 are generically perturbed *inside* U^{p^W}. To complete our proof, it is enough to turn these perturbations into analytic ones by using the method employed in the proof of Proposition (5.4). The lemma is proved. □

Proof of Proposition (7.2). Let us consider the vector field Y^2 given by Lemma (7.3). In the linearizing coordinate ϕ which identifies p with the origin of $\mathbb{R}^n$, we set $Y^2(0) = (v_1, \dots, v_n)$ with $v_1 \dots v_n \neq 0$. Next recall that $F^s(x_1, \dots, x_n) = (\lambda_1 x_1, \dots, \lambda_n x_n)$ where $0 <| \lambda_1 |< \cdots <| \lambda_n |$. Modulo multiplication by a suitable sequence of scalars, the sequence of vector fields $(F^{sk})^* Y^2$ ($k \in \mathbb{N}$) clearly converges to the constant vector field $\partial/\partial x_n$. Next combining Y^2 and $\partial/\partial x_n$, we obtain a vector field Y^3 in the C^∞-closure of G such that $Y^3(0) = (v_1, \dots, v_{n-1}, 0)$. Repeating the procedure, we see that appropriate scalar multiples of the sequence $(F^{sk})^* Y^3$ converge to the constant vector field $\partial/\partial x_{n-1}$. Continuing inductively this procedure, we eventually conclude that all the constant vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$ belong to the C^∞-closure of G. Proposition (7.2) is proved. □

In view of Proposition (7.2), there exists a residual set $\mathcal{K}_2 \subset \mathcal{K}_1$ such that the group G generated by f_1, f_2 verifies the conclusions of this proposition provided that (f_1, f_2) belongs to $\mathcal{K}_2$. To set up the proof of Theorem D, the last needed ingredient is Proposition (7.4).

Proposition 7.4. *Let (f_1, f_2) and $(\hat{f}_1, \hat{f}_2)$ be pairs of diffeomorphisms in $\mathcal{K}_2$ and denote by G (resp. $\widehat{G}$) the group generated by f_1, f_2 (resp. $\hat{f}_1, \hat{f}_2$). Suppose that h is a homeomorphisms conjugating G_1 and G_2 (i.e. $\mathsf{h} \circ f_1 \circ \mathsf{h}^{-1} = \hat{f}_2$, $\mathsf{h} \circ f_2 \circ \mathsf{h}^{-1} = \hat{f}_2$). Then h is, in fact, an element of* $\mathrm{Diff}^{\omega}(M)$.

Complement to the proof of Theorem D. Observe that any point $q \in M$ possesses a neighborhood endowed with n vector fields $X_{1,q}, \ldots, X_{n,q}$ which are linearly independent at q (where n stands for the dimension of M). In fact, according to Proposition (7.2), there exist linearly independent vector fields $X_{1,p}, \ldots, X_{n,p}$ on a small neighborhood of any point p which is either an attractor or a repeller of some F^s ($s \in \mathbb{N}$, F a Morse-Smale diffeomorphism belonging to G and satisfying the preceding assumptions). On the other hand, Lemma (7.1) ensures that the G-orbit of any point $q \in M$ intersects the union of the neighborhoods considered above. This shows the existence of the desired vector fields $X_{1,q}, \ldots, X_{n,q}$ in the closure of G and linearly independent at q. Using the argument already discussed in Section 4, it follows that G has all orbits dense and is ergodic with respect to the Lebesgue measure.

Finally the fact that homeomorphisms conjugating groups as above are, indeed, analytic diffeomorphisms is precisely the content of Proposition (7.4). The proof of the Theorem D is over. $\square$

It still remains to prove Proposition (7.4). Let $F \in G$ be a Morse-Smale diffeomorphism satisfying the conditions in the beginning of the present section. Again consider the points $p_1, \ldots, p_r$ in $\Omega(F)$ of periods $s_1, \ldots, s_r$ such that p_i is either a hyperbolic attractor or a hyperbolic repeller for F^{s_i}. Finally let us associate a neighborhood U_i to each of the p_i's.

Now consider the group $\widehat{G}$ and the homeomorphism h. Clearly the points $\mathsf{h}(p_1), \ldots, \mathsf{h}(p_r)$ are either hyperbolic attractors or hyperbolic repellers for $\hat{F}^{s_i}$, where $\hat{F} = \mathsf{h} \circ F \circ \mathsf{h}^{-1} \in \widehat{G}$ (here we use Condition 5). Furthermore, thanks to Lemma (7.1), the $\widehat{G}$-orbit of any point $q \in M$ intersects $\bigcup_{i=1}^{r} \mathsf{h}(U_i)$. Therefore, to prove Proposition (7.4), it is enough to check that h coincides with an analytic diffeomorphism on, say, a neighborhood U_1 of p_1. Without loss of generality, we can suppose that p_1 is a hyperbolic attractor of F^{s_1} (so that $\mathsf{h}(p_1)$ is a hyperbolic attractor of $\hat{F}^{s_1}$).

Lemma 7.5. *If U_1 is chosen sufficiently small, then there exists a non-trivial vector field X (resp. $\hat{X}$) defined on U_1 (resp. $\mathsf{h}(U_1)$) and contained in the* *closure of G (resp. $\widehat{G}$). Furthermore h realizes a local conjugacy between the local flows Φ_X, $\hat{\Phi}_{\hat{X}}$ of X, $\hat{X}$ in the sense that the equation*

$$\mathsf{h} \circ \Phi_X^t \circ \mathsf{h}^{-1}(q) = \hat{\Phi}_{\hat{X}}^t(q) \tag{13}$$

holds provided that both members are defined.

Proof. To prove this lemma we need to revisite the proof of Proposition (2.1). We keep the corresponding notations. Using local charts around $p, \mathsf{h}(p)$, we can suppose that these points are identified with the origin of $\mathbb{C}^n$ (while h is defined only on $\mathbb{R}^n \subset \mathbb{C}^n$). Denote by $\Gamma, \widehat{\Gamma}$ the pseudogroups induced by $G, \widehat{G}$ on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ through the corresponding local charts. We shall identify elements of $G, \widehat{G}$ with elements of $\Gamma, \widehat{\Gamma}$ in the obvious way. Finally, under these identifications, F (resp. $\hat{F}$) is given as a contractive homothety $(z_1, \ldots, z_n) \mapsto (\lambda_1 z_1, \ldots, \lambda_n z_n)$ (resp. $(z_1, \ldots, z_n) \mapsto (\hat{\lambda}_1 z_1, \ldots, \hat{\lambda}_n z_n)$).

We first notice the existence of sequences $\{h_i\} \subset \Gamma$, $\{\hat{h}_i\} \subset \widehat{\Gamma}$ which converge uniformly to the identity on $\mathbb{B}^n$ and verifies $\mathsf{h} \circ h_i \circ \mathsf{h} = \hat{h}_i$. Actually to obtain these sequences it suffices to consider sequences of iterated commutators as in Theorem (5.2).

Next we fix a monotone decreasing sequence $\{\delta_r\}$ of positive reals converging to *zero*. Again we consider the elements of Γ given by $F^{-1} \circ h_i \circ F$, $\cdots$, $F^{-j(i)} \circ h_i \circ F^{j(i)}$ where $j(i)$ is the greatest positive integer for which the map $F^{-j(i)} \circ h_i \circ F^{j(i)}$ is defined on $B(4\lambda_1/5)$.

Recall that h is defined only on $\mathbb{B}^n \cap \mathbb{R}^n$. Let us denote $\mathbb{V}_{3/4}$ and $\mathbb{V}_{1/2}$, $\overline{\mathbb{V}}_{1/2} \subset \mathbb{V}_{3/4}$, open sets of $\mathbb{C}^n$ such that $\mathbb{V}_{3/4} \cap \mathsf{h}(\mathbb{B}(3\lambda_1/4) \cap \mathbb{R}^n)$ and $\mathbb{V}_{1/2} \cap \mathsf{h}(\mathbb{B}(\lambda_1/2) \cap \mathbb{R}^n)$ (without loss of generality we can suppose that $\overline{\mathbb{V}}_{3/4} \subset \mathbb{B}^n$). Consider also an open set $\mathbb{V}_{4/5} \subset \mathbb{B}^n$ which strictly contains $\overline{\mathbb{V}}_{3/4}$.

Next we shall also consider the elements of $\widehat{\Gamma}$ given by $\hat{F}^{-1} \circ \hat{h}_i \circ \hat{F}, \cdots,$ $\hat{F}^{-\hat{j}(i)} \circ \hat{h}_i \circ \hat{F}^{\hat{j}(i)}$ where $\hat{j}(i)$ is the greatest positive integer for which the map $\hat{F}^{-\hat{j}(i)} \circ \hat{h}_i \circ \hat{F}^{\hat{j}(i)}$ is defined on $\mathbb{V}_{4/5}$.

In the sequel we are going to discuss only the case $j(i) < \infty$, $\hat{j}(i) < \infty$ for every i. The other possibilities are analogous and left to the reader. Fix $\delta_r > 0$ and, for each i, define $j(i,r)$ (resp. $\hat{j}(i,r)$) as the smallest positive integer for which $\sup_{\mathbb{B}(3\lambda_1/4)} \| F^{-j(i,r)} \circ h_i \circ F^{j(i,r)}(z) - z \| > \delta_r$ (resp. $\sup_{\mathbb{V}_{3/4}} \| \hat{F}^{-\hat{j}(i,r)} \circ \hat{h}_i \circ \hat{F}^{\hat{j}(i,r)}(z) - z \| > \delta_r$). Now, recalling that r was already fixed, we choose and fix i very large. Let $J(i,r)$ be the minimum between $j(i,r)$, $\hat{j}(i,r)$ and $C(i,r)$ the maximum between $\sup_{\mathbb{B}(3\lambda_1/4)} \| F^{-J(i,r)} \circ h_i \circ F^{J(i,r)}(z) - z \|$, $\sup_{\mathbb{V}_{3/4}} \| \hat{F}^{-J(i,r)} \circ \hat{h}_i \circ \hat{F}^{J(i,r)}(z) - z \|$.

Consider the sequences $\{X_r\}$, $\{\hat{X}_r\}$ of vector fields defined respectively on $\mathbb{B}(3\lambda_1/4)$, $\mathbb{V}_{3/4}$, where

$$\begin{aligned} X_r &= \frac{1}{C(i,r)} \operatorname{Vect}\,(F^{-J(i,r)} \circ h_i \circ F^{J(i,r)}(z) - z)\,, \\ \hat{X}_r &= \frac{1}{C(i,r)} \operatorname{Vect}\,(\hat{F}^{-J(i,r)} \circ \hat{h}_i \circ \hat{F}^{J(i,r)}(z) - z)\,. \end{aligned}$$

Clearly both sequences of vector fields are uniformly bounded on their domains. Thus we can find limits X and $\hat{X}$ which are defined respectively on $B(\lambda_1/2)$ and $\mathbb{V}_{1/2}$. The local flows of these vector fields naturally verifies Equation (13). Finally if, for each previously fixed r, i was chosen very large, we have ensured that at least one between X, $\hat{X}$ is not the trivial vector field. However in this case Equation (13) shows that both vector fields are in fact non-trivial since they have real coefficients. This proves the lemma. □

Finally we have

Proof of Proposition (7.4). We consider vector fields X, $\hat{X}$ satisfying Equation (13). Using the argument employed in Lemma (7.3), we can suppose that $X(0) \neq 0$. Thus Equation (13) yields $\hat{X}(0) \neq 0$ as well.

We now recall that F^{s_1} (resp. $\hat{F}^{s_1}$) coincides in appropriate local charts with the homothety $(z_1, \ldots, z_n) \mapsto (\lambda_1 z_1, \ldots, \lambda_n z_n)$ (resp. $(z_1', \ldots, z_n') \mapsto (\hat{\lambda}_1 z_1', \ldots, \hat{\lambda}_n z_n')$). Besides one has $0 <\mid \lambda_1 \mid< \cdots <\mid \lambda_n \mid< 1$ (resp. $0 <\mid \hat{\lambda}_1 \mid< \cdots <\mid \hat{\lambda}_n \mid< 1$). Thus by iterating conjugations of X, $\hat{X}$ as in the proof of Proposition (7.2), we deduce that h realizes a conjugacy (i.e. a time-preserving equivalence) between the pairs $\partial/\partial z_i$ and $\partial/\partial z_i'$ $(i = 1, \ldots, n)$ of vectors fields. It follows that h coincides with the identity in these local coordinates. The proof of Proposition (7.4) is completed. □

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